Lecture Notes for ECG 742, “Consumption, Demand, and Market Interdependencies”*

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I. Introduction

A Brief Overview of Developments in Demand

Demand analysis historically generally has followed two paths: (1) discovery of general laws governing operations of markets (particularly agricultural markets), and (2) work of those people interested in psychological laws governing consumer preferences.

Historical precedence for demand originated in the work of Davenant (1699), who published a numerical schedule of the demand for wheat. In the 1700s, Verri and Floyd were the first to attempt to sort out the separate influences of demand and supply on market price.

In 1730, Bernoulli laid foundations for consumer demand by proposing a logarithmic utility function. In the late 1800s, there was considerable focus by economists to provide an axiomatic foundation for the model of market equilibrium suggested by Adam Smith. Economists pursuing this work included Gossen, Jevons, Walras, and Edgeworth. Edgeworth and Pareto's work led to establishment of modern theory based on ordinal rather than cardinal utility.

Throughout the 18^{th} and 19^{th} centuries, there was little progress in measurement of demand curves. Instead, the focus was on the influence of income on consumption. In 1857, Engel formulated what have now become enduring empirical laws.

In the late 19^{th} century, both theoretical and empirical approaches focused on writings of Marshall. Marshall encouraged agricultural economists to apply newly discovered correlation analysis to analysis of demand in single markets. Marshall also clarified and elaborated on the concept of elasticity of demand, which has become so central to both theoretical and empirical analysis in demand.

It is no accident that agricultural commodities were the initial focus of economists in empirical analysis. This is because of important conditions characterizing these markets: (1) homogenous commodities, (2) stable consumer preferences, and (3) relatively large fluctuations in supply independent of current market price. In 1907, Benini used multiple regression analysis to estimate a demand function for coffee in Italy with prices of coffee and sugar as explanatory variables in the regression equation. In 1914, Lehfeldt, under supervision of Marshall, estimated a demand elasticity for wheat in England.

The most serious progress in econometrics of demand was achieved by agricultural economists in the United States, beginning with Moore who published many studies between 1914 and 1929. Moore's work stimulated work on estimation techniques and relation between correlation analysis and causal models. Developments in these areas were made by Yule, Hotelling, Working, Schultz, Ezekiel, and Leontief. Also, at this time, Allen and Hicks (1934) independently rediscovered the Slutsky decomposition of demand into substitution and income effects. All this work firmly established the theory of demand among English-reading economists and econometricians. Indeed, by 1939, most of the strengths and weaknesses of classical demand analysis, in the context of single-equation fitting to both time-series and cross-section data, had been discovered.

The main problems with this approach are as follows:

(a) Assumption of constant elasticities: nations become richer over time and income elasticities become smaller or larger as income changes. If income elasticities are constant, eventually the goods whose (constant) income elasticities are larger than one will consume all the budget and expenditures on these goods will exceed total income.

(b) Per capita demand depends on distribution of income: characteristics of individuals unlikely to be reproduced in the aggregate and, therefore, have to rely on assumption that "representative" consumer is somewhere represented in the population.

(c) Absence of money illusion as a maintained hypothesis.

(d) Problem of making provision for influence of competing goods on consumption of isolated commodities.

Since these early days of pioneering work in demand, economists have focused both theoretically and empirically on systems of demand functions, durable goods and dynamics, and improved computational methods. Samuelson and Houthakker's work on revealed preference theory was significant to this development and, in particular, the work of Samuelson which led to solution of the integrability problem in consumer demand. Prior to the middle 1960s, estimation of elasticities was the main focus of economists. Since that time, testing of theory has become the additional main area of interest.

<u>Nature of Concerns of Demand Analysts</u>

Demand analysts have traditionally been concerned with how demand for a given commodity will be altered as certain specified variables change. The primary types of variables of interest include: (1) changes in real income per capita on per capita consumption, (2) structure of relative commodity prices, (3) distribution of income, and (4) introduction of new commodities and changes in tastes and preferences.

The earliest approach to demand analysis was coined the "pragmatic approach" by Stone and others in the 1950s. In this approach, one selects a particular functional form (e.g., double logarithmic functional form) such that

$$\log q_i = a_i + b_i t + e_i \log(x/\pi) + e_{ii} \log(p_i/\pi) \qquad (1)$$

where q_i, p_i represent per capita consumption and price of the ith commodity, respectively; and x, π represent per capita income and general price level, respectively. The variable *t* is a linear time trend and the coefficients associated with the variables are assumed to be constant. Since the equation is linear in logarithms the coefficients associated with per capita real income and real own price are income elasticity , e_i , and own-price elasticity of demand, e_{ii}, respectively.

Although adequate for many purposes, one may raise several questions about the adequacy of equation (1) for empirical analysis. First, what is the basis for selecting the double logarithmic functional form and is there another functional form that would be preferable? Second, although the influence of other prices can be taken into account through introduction of a general price variable as deflator for per capita income and own-price, is this the best way to remove money illusion? In other words, is the appropriate and best way to account for the influence of prices of other goods? Finally, equation (1) indicates that all other changes to demand can be accounted for by a linear trend variable. This may not be wholly adequate owing to the existence of dynamics associated with durable goods and introduction of new commodities. These are only some of the many questions the analyst might want to ask about the appropriate specification of consumer demand for a given commodity.

The discussion so far has taken as the object of study final consumers of the commodity. In many instances, however, the data or questions of interest are such that derived demand rather than consumer demand is the main focus. The simplest case envisioned would be derived demand for a factor of production. For example, interest might be in really demand for cattle (an input into beef production) rather than demand for beef per se. The more complicated case would be derived demand for a good in the context of household production theory, where final goods are purchased with commodities and other inputs (e.g., time) so that the consumer maximizes utility from consumption of unobservable final goods. The theory of derived demand suggests that there are many types of demand functions to consider: output constant demands, output demand constant, and output price constant to mention the main types. How do we obtain these different derived demand functions and how are they related to final consumer demand for the commodities?

To Estimate or Forecast?

The goal of demand analysis is frequently to estimate income and price elasticities. However, this may be only one of the objectives of the analysis and another purpose might be to use the elasticities to come up with conditional forecasts of the expected quantities and prices under certain specified conditions. It is important to take such considerations into account in the empirical analysis because the ultimate use of the model can have consequences for estimation as the following example illustrates.

Consider the analysis of determining the effect of raising taxes on cigarettes on tax revenue collected from alcoholic beverages. In this context, our objective would be to obtain an estimate of the cross-price elasticity of demand for alcoholic beverages with respect to a change in the price of cigarettes. Such a study would require that we estimate the two demand functions, demand for cigarettes and demand for alcoholic beverages, jointly. We would also want the cross-price elasticities of each commodity be estimated in a consistent way so, as we shall see later, certain cross-equation restrictions would need to be imposted in estimation to ensure that the demand functions for cigarettes and alcoholic beverages are consistent with one another.

Suppose as another example that you are an economic consultant hired to evaluate the effectiveness of generic beef advertising on the U.S. beef industry. You postulate the following model of the beef industry:

$$q_b^d = D_b(p_b, p_p, a) \qquad \text{(Demand for beef)}$$

$$q_b^d = D_b(p_b, p_p, a) \qquad \text{(Demand for pork)}$$

$$q_b^s = S_b(P_b, a) \qquad \text{(Supply of beef)}$$

$$q_p^s = S_p(P_p) \qquad \text{(Supply of pork)}$$

where $q_b^d q_b^s, q_p^d, q_p^s$ represent quantities of beef (b) and pork (p) demanded (d) and supplied (s); p_b, p_p represent prices of beef and pork, respectively; and *a* represents generic advertising of beef. For sake of convenience, we ignore the interaction with other commodities like poultry and generic advertising in pork.

To develop the comparative statics of a change in advertising on the beef industry, we must take into account the interaction of beef with pork in both demand and supply. Focusing on the effect on the price of beef we equate quantity supplied with quantity demanded in each market to obtain the two equation system:

$$S_b(P_b, a) = D_b(P_b, P_p, a)$$
$$S_p(P_p) = D_p(P_b, P_p, a)$$

Totally differentiating this system and solving for $\frac{dp_b}{da}$ gives

$$\frac{dp_b}{da} = \frac{(\partial S_p / \partial p_p - \partial D_p / \partial p_p)(\partial D_b / \partial a - \partial S_a / \partial a) + (\partial D_b / \partial p_p)(\partial D_p / \partial a)}{[(\partial S_b / \partial p_b - \partial D_b / \partial p_b)(\partial S_p / \partial p_p - \partial D_p / \partial p_p) - (\partial D_b / \partial p_p)(\partial D_p / \partial p_b)} \qquad (2)$$

Although this derivative appears quite complicated it should be clear that the impact of generic advertising of beef on beef price cannot be evaluated without information on the effects of beef and pork prices on both demand and supply of beef and demand and supply of pork. Of particular significance is that knowledge of own-price and cross-price effects is necessary in order to determine the value of the derivative in equation (2). Therefore, estimation of demand—even when one is only interested in one isolated commodity—often requires that the demand system be estimated jointly.

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II. Consumer Demand Analysis

Properties of Demand Functions

Without considering preferences and by assuming the consumer faces a linear budget constraint, it is possible to derive some fundamental properties of demand functions. Assume that there exist Marshallian demand functions:

$$q_i = g_i(x, p) \qquad i=1,2,\ldots n,$$

where $p = (p_1, p_2, ..., p_n)'$ is vector of commodity prices. Substituting these demand functions into the linear budget constraint, $x = \sum_k p_k q_k$, yields the following results:

$$x = \sum_k p_k g_k(x, p) \qquad \text{(adding-up property)} \qquad (1)$$

$$g_i(\theta x, \theta p) = g_i(x, p), \theta > 0, \forall i \qquad \text{(homogeneity)} \qquad (2)$$

It is sometimes useful to express these results in elasticity form. Differentiating equation (1) with respect to x and p_i:

$$1 = \sum p_k \frac{\partial g_k}{\partial x}; \qquad 0 = \sum p_k \frac{\partial g_k}{\partial p_i} + q_i$$

which are the *Engel Aggregation* and *Cournot Aggregation* conditions, respectively.

By Euler's theorem, condition (2) above can be written as

$$\sum p_k \frac{\partial g_i}{\partial p_k} + x \frac{\partial g_i}{\partial x} = 0 \qquad \text{(homogeneity restriction)}$$

Let $w_i = \frac{p_i q_i}{x}, e_{ij} = \frac{p_j}{q_i}\frac{\partial g_i}{\partial p_j} = \frac{\partial(\log q_i)}{\partial(\log p_j)}, e_i = \frac{x}{q_i}\frac{\partial g_i}{\partial x} = \frac{\partial(\log q_i)}{\partial(\log x)}$. Therefore, we can express the Engel, Cournot, and homogeneity conditions in elasticity form as follows:

$$1 = \sum_k w_k e_k \qquad \text{(Engel aggregation)} \qquad (3)$$

$$0 = \sum_k w_k e_{ki} + w_i, \quad \forall i \qquad \text{(Cournot aggregation)} \qquad (4)$$

$$\sum_k e_{ik} + e_i = 0, \quad \forall i \qquad \text{(Euler aggregation)} \qquad (5)$$

Consider the double log demand model:

$$\log q_i = \alpha_i + e_i \log x + \sum_k e_{ik} \log p_k + u_i$$

With appropriate data, we could apply OLS to this equation to obtain estimates of the income and price elasticities. Homogeneity can be imposed and tested, but the adding-up property will not hold unless all the income elasticities equal unity. Why? From the budget share we find that

$$\begin{aligned} \log w_i &= \log q_i + \log p_i - \log x \\ &= \alpha_i + (e_i - 1)\log x + (e_{ii} + 1)\log p_i \\ &\quad + \sum_{k \neq i} e_{ik} \log p_k \end{aligned}$$

If income and prices change, the expenditure share of the *i*th good will change causing the Engel aggregation condition, equation (3), to be violated. In order for w_i to be constant for arbitrary changes in x and p:

$$e_i = 1, e_{ii} = -1, e_{ik} = 0, \qquad \forall k \neq i$$

This also means that each demand function must have the special form

$$p_i q_i = \beta_i x$$

which necessarily implies that the underlying preferences are Cobb-Douglas.

Remark: while the double log demand model is compatible with the Euler aggregation condition (homogeneity condition), it fails to satisfy the adding-up condition except under certain (rather restrictive) conditions.

Despite this limitation of the double log demand model, it still can be useful as a single-equation model because such a specification does not commit us to the double log functional form for other commodities. Thus, the model can be valid for limited variation in x for *incomplete* demand systems.

Preferences and Demand

Usually we postulate that the consumer's choice problem is to

$$\max_{q} v(q)$$

$$\text{subject to:} \quad \sum_k p_k q_k \leq x,$$

$$q_k \geq 0, \quad \forall k$$

The utility indicator or function, $v(\bullet)$, is assumed to be a strictly quasi-concave, monotonic increasing function. That is, it is assumed to be an ordinal indicator of preferences over a given set of n commodities. The utility function is derived from the following set of axioms:

1. reflexivity: $q \succeq q$
2. completeness: either $q^1 \succeq q^2$ or $q^2 \succeq q^1$
3. transitivity: if $q^1 \succeq q^2, q^2 \succeq q^3$, then $q^1 \succeq q^3$
4. continuity: if $A(q^1) = \left\langle q \middle| q \succeq q^1 \right\rangle$--"at least as good as q^1 set";

   $B(q^1) = \left\langle q \middle| q^1 \succeq q \right\rangle$--"no better than q^1 set"; then

   then $A(q^1)$ and $B(q^1)$ are closed sets (i.e., they contain their own boundaries.)

Axioms 1-4 imply that we can represent the preference ordering by a single-valued utility function. That is, $q^1 \succeq q^2 \Leftrightarrow v(q^1) \geq v(q^2)$.

Two additional axioms are often added. These are

5. Nonsatiation: $v(q)$ is non-decreasing in each element and increasing in at least one element.
6. Convexity: if $q^1 \succeq q^2$ then for $0 \leq \lambda \leq 1$, $\lambda q^1 + (1-\lambda)q^2 \succeq q^2$. Equivalently, $A(q^1)$ is a convex set.

Axiom 6 implies that $v(q)$ is quasi-concave, which in turn means indifference curves will show diminishing marginal rate of substitution. (Q: Does diminishing MRS imply diminishing MU?)

Kuhn-Tucker Theorem:

If q^* solves the utility maximization problem, subject to a constraint qualification, then there exists a non-negative scalar λ such that

$$\frac{\partial v(q^*)}{\partial q_i} \leq \lambda p_i, \qquad \forall i$$

$$\sum p_k q_k^* \leq x$$

$$\left[\frac{\partial v(q^*)}{\partial q_i} - \lambda p_i\right] q_i^* = 0, \qquad \forall i$$

$$\left[\sum p_k q_k^* - x\right]\lambda = 0$$

Clearly, if $\frac{\partial v(q^*)}{\partial q_i} < \lambda p_i$, then $q_i^* = 0$ and if

$\sum p_k q_k^* < x$ then $\lambda = 0$.

Given the axioms of continuity and nonsatiation (axioms 4 and 5) the latter equality will always hold. That is, the budget will always be fully spent. With the assumption that $v(q)$ is strictly quasi-concave the K-T conditions above are also sufficient.

Marshallian demand functions:

If there are no corner solutions and the budget constraint is binding, then the F.O.N.C above can be written as:

$$\frac{\partial v(q^*)}{\partial q_i} = \lambda p_i, \qquad \forall i$$

$$\sum p_k q_k^* = x$$

These F.O.N.C. represent n+1 equations in n+1 unknowns (q^*, λ). The solutions are the *Marshallian demand functions*:

$$q_i = g_i(x, p), \qquad \forall i$$

What are the properties of these functions? We have already derived some of those properties which we will derive again. We can ascertain the properties of these demand functions in two ways: through direct differentiation of the F.O.N.C. above or through the use of duality theory.

Dual Approach

Original problem:

$$\max_q \; u = v(q)$$

s.t. $p \bullet q = x$

Dual problem:

$$\min_q \; p \bullet q = x$$

s.t. $u = v(q)$

Solutions to original problem are Marshalllian demands $q_i = g_i(x, p)$; solutions to dual problem are Hicksian demands $q_i = h_i(u, p)$. Solutions to two problems will be the same when *u* in dual problem is maximized value of utility in original problem and when minimum value of *x* from dual problem is value of *x* in original problem.

If these conditions hold, then

$$q_i = g_i(x, p) = h_i(u, p) \qquad \forall i .$$

If we substitute the Marshallian demands into the direct utility function we obtain

$$u = v[g_1(x, p), g_2(x, p), ..., g_n(x, p)] = \psi(x, p)$$

The indirect utility function—the maximum attainable utility with *x* and *p*.

If we substitute the Hicksian demands into the budget constraint we obtain

$$x = \sum p_k h_k(u, p) = c(u, p)$$

The expenditure (cost) function—the minimum attainable cost given *u* and *p*.

We also could represent these relationships as follows:

$$\psi(x, p) = \max_q [v(q) : p \bullet q = x]$$

$$c(u, p) = \min_q [p \bullet q : v(q) = u]$$

Note: inversion of the indirect utility function gives the expenditure function and vice versa.

Properties of cost function:

1. Homogenous of degree one in prices, i.e.,
   $c(u, \theta p) = \theta c(u, p), \theta > 0$
2. Cost function is increasing in *u*, non-decreasing in *p*, and increasing in at least one p_i.
3. Concave in prices.
4. Continuous in *p*.
5. Derivative property (Shephard's lemma): $\frac{\partial c}{\partial p_i} = h_i(u, p) = q_i$.

Relationship between Marshallian and Hicksian demands:

$$q_i = h_i[\psi(x,p),p] = g_i(x,p)$$

Shephard's lemma allows us to link the two types of demand functions. Another way to link the two demand functions is *Roy's identity*:

$$q_i = g_i(x,p) = \frac{-\partial\psi(x,p)/\partial p_i}{\partial\psi(x,p)/\partial x}.$$

Properties of consumer demand functions:

1. Adding-up: $\sum p_k g_k(x,p) = \sum p_k h_k(u,p) = x$.
2. Homogeneity: $h_i(u,\theta p) = h_i(u,p); g_i(\theta x,\theta p) = g_i(x,p)$.
3. Symmetry: for all $i \neq j$, $\frac{\partial h_i}{\partial p_j} = \frac{\partial h_j}{\partial p_i}$.
4. Negativity: nxn matrix $[\partial h_i/\partial p_j]$ is negative semi-definite.
5. Slutsky equation: $\frac{\partial h_i}{\partial p_j} = \frac{\partial g_i}{\partial p_j} + \frac{\partial g_i}{\partial x} q_j$.

<u>Fundamental Demand Matrix</u>

The F.O.N.C. for utility maximization in matrix notation are

$$\begin{aligned} v_q &= \lambda p \\ p'q &= x \end{aligned}$$

where $v_q = \partial v/\partial q = [v_1, v_2, ..., v_n]', p = [p_1, p_2, ... p_n]'$. The solutions to this problem are

$$\begin{aligned} q &= g(x,p) \\ \lambda &= \lambda(x,p) \end{aligned}$$

The differential of vector of marginal utilities is

$$U = \left[\frac{\partial^2 v}{\partial q_i \partial q_j} \right] = \left[v_{ij} \right]$$

Total differentiation of the FONC yields:

$$\begin{aligned} Udq &= pd\lambda + \lambda dp \\ dx &= p'dq + q'dp \end{aligned}$$

or
$$\begin{bmatrix} U & p \\ p' & 0 \end{bmatrix} \begin{bmatrix} dq \\ -d\lambda \end{bmatrix} = \begin{bmatrix} 0 & \lambda I \\ 1 & -q' \end{bmatrix} \begin{bmatrix} dx \\ dp \end{bmatrix} \tag{1}$$

Total differentiation of the Marshallian demand functions and $\lambda = \lambda(x, p)$ gives

$$\begin{bmatrix} dq \\ -d\lambda \end{bmatrix} = \begin{bmatrix} q_x & Q_p \\ -\lambda_x & -\lambda_p' \end{bmatrix} \begin{bmatrix} dx \\ dp \end{bmatrix} \tag{2}$$

where $q_x = \left[\frac{\partial g_i}{\partial x} \right], \lambda_x = \frac{\partial \lambda}{\partial x}, \lambda_p = \left[\frac{\partial \lambda}{\partial p_i} \right], Q_p = \left[\frac{\partial g_i}{\partial p_j} \right]$. Substituting (2) into (1) we see that (1) can be written as follows:

$$\begin{bmatrix} U & p \\ p' & 0 \end{bmatrix} \begin{bmatrix} q_x & Q_p \\ -\lambda_x & -\lambda_p' \end{bmatrix} = \begin{bmatrix} 0 & \lambda I \\ 1 & -q' \end{bmatrix} \tag{3}$$

which is Barten's *Fundamental Demand Equation.*

The solution to (3), in terms of q_x and Q_p requires inversion of the left-hand side matrix. Provided *v(q)* is strictly quasi-concave the LHS matrix will be non-singular and unique solutions exist. If we also were to assume that U is non-singular we could apply the rule for inversion of a partitioned matrix to obtain

$$\begin{bmatrix} U & p \\ p' & 0 \end{bmatrix}^{-1} = (p'U^{-1}p)^{-1} \begin{bmatrix} (p'U^{-1}p)U^{-1} - U^{-1}pp'U^{-1} & U^{-1}p' \\ p'U^{-1} & -1 \end{bmatrix} = R$$

Therefore

$$\begin{bmatrix} q_x & Q_p \\ -\lambda_x & -\lambda_p' \end{bmatrix} = R \begin{bmatrix} 0 & \lambda I \\ 1 & -q' \end{bmatrix}$$

so that

$$\begin{aligned} Q_p &= \lambda U^{-1} - \lambda(p'U^{-1}p)^{-1}U^{-1}pp'U^{-1} - (p'U^{-1}p)^{-1}U^{-1}pq' \\ q_x &= (p'U^{-1}p)^{-1}U^{-1}p \\ -\lambda_p' &= \lambda(p'U^{-1}p)^{-1}p'U^{-1} + q' \\ -\lambda_x &= -(p'U^{-1}p)^{-1} \end{aligned}$$

The expression for Q_p can be simplified

$$\begin{aligned} Q_p &= \lambda U^{-1} - \lambda q_x p'U^{-1} - q_x q = \lambda U^{-1} - \lambda q_x (p'U^{-1}p)(p'U^{-1}p)^{-1} p'U^{-1} - q_x q' \\ &= \lambda U^{-1} - \lambda\lambda_x^{-1} q_x q_x' - q_x q' \end{aligned}$$

Hence,

$$S = \lambda U^{-1} - \lambda\lambda_x^{-1} q_x q_x' \qquad (4)$$

which is the *Slutsky Substitution Matrix.*

This exposition has the advantage of showing that the substitution effect can be decomposed into two effects:

First component: λU^{ij} is the specific substitution effect

Second component: $-\lambda\lambda_x^{-1}\frac{\partial g_i}{\partial x}\frac{\partial g_j}{\partial x}$ is the general substitution effect.

Specific and general substitution effects:

The specific and general substitution effects from a compensated price change result from holding the marginal utility constant. In particular, *the specific substitution effect* shows how much quantity would change as price changes holding the marginal utility of income constant. *The general substitution effect* is the effect on quantity (from a price change) of a change in real income holding the marginal utility constant.

To see how these effects can be interpreted as such, consider the total differential of the marginal utility of income: $d\lambda = \boldsymbol{\lambda_p'dp} + \lambda_x dx$. If all prices but the jth price are held constant, the total differential is $d\lambda = \lambda_{p_j}dp_j + \lambda_x dx$. When the marginal utility of income is constant, $d\lambda = 0$, and $dx = -\frac{\lambda_{p_j}}{\lambda_x}\,dp_j$. From the results for the *Fundamental Demand Matrix*,

$\frac{\partial\lambda}{\partial p_j} = -\lambda\frac{\partial q_j}{\partial x} - \lambda_x q_j$. Therefore, the income compensation required to hold the marginal utility constant for a change in p_j is as follows: $dx = \frac{\lambda}{\lambda_x}\frac{\partial q_j}{\partial x}dp_j + q_j dp_j$.

Using this result the effect on demand for good i is $\frac{\partial q_i}{\partial p_j}dp_j +$

$\frac{\partial q_i}{\partial x}dx$ (holding λ constant) $= \frac{\partial q_i}{\partial p_j}dp_j + \frac{\lambda}{\lambda_x}\frac{\partial q_i}{\partial x}\frac{\partial q_j}{\partial x}dp_j + q_j\frac{\partial q_i}{\partial x}dp_j$.

Therefore, when $dp_j = 1$, the effect on demand is $\frac{\partial q_i}{\partial p_j} + \frac{\lambda}{\lambda_x}\frac{\partial q_i}{\partial x}\frac{\partial q_j}{\partial x} + q_j\frac{\partial q_i}{\partial x}$. But we know from the decomposition of the substitution effect from the *Fundamental Demand Matrix* that

$\frac{\partial q_i}{\partial p_j} + \frac{\lambda}{\lambda_x}\frac{\partial q_i}{\partial x}\frac{\partial q_j}{\partial x} + q_j\frac{\partial q_i}{\partial x} = U^{ij}$, which is *the specific substitution effect.*

Note from above that $dx - q_j dp_j = \frac{\lambda}{\lambda_x}\frac{\partial q_j}{\partial x}dp_j$. The effect on q_i from a change in real income necessary to hold λ constant is $\frac{\partial q_i}{\partial x}(dx - q_j dp_j) = \frac{\partial q_i}{\partial x}\left(\frac{\lambda}{\lambda_x}\frac{\partial q_j}{\partial x}dp_j\right)$. Thus, when $dp_j = 1$, *the general substitution effect* is $\frac{\lambda}{\lambda_x}\frac{\partial q_i}{\partial x}\frac{\partial q_j}{\partial x}$.

<u>Distance Functions</u>

The distance function can be used to give an inverse representation of the direct utility function.

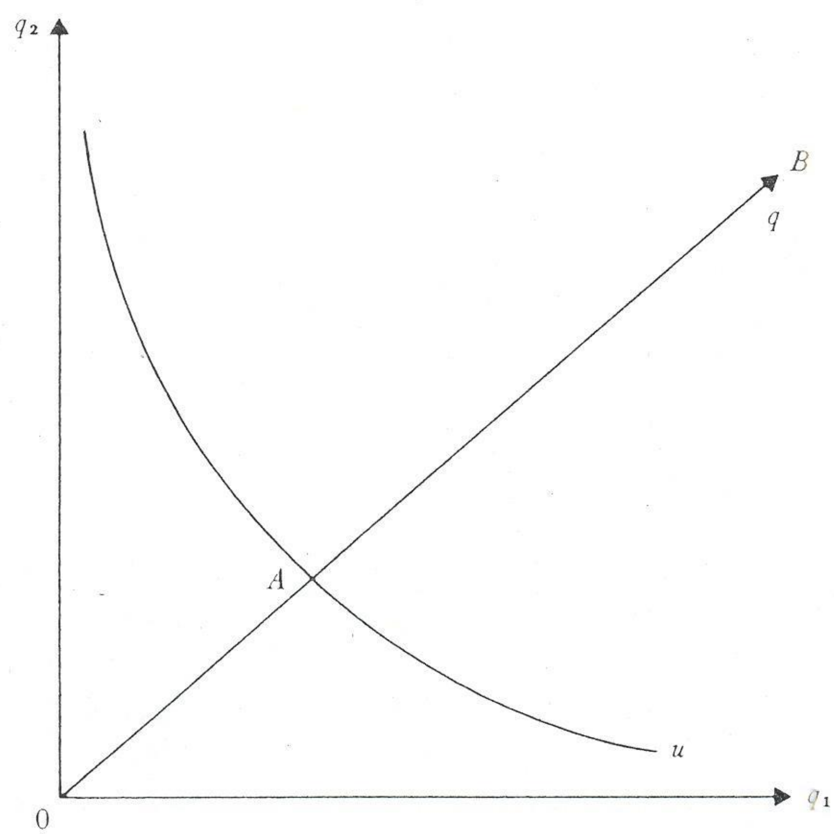

Figure 1 . Distance Function

The distance function, *d(u,q)*, is the amount by which *q* must be divided in order to bring it onto the curve *u*, i.e.,

$$d(u,q) = OB/OA$$
$$\Rightarrow v[q/d(u,q)] = u$$

If *q* lies on *u* subject to the restriction *u=v(q)*, then *OB=OA.* Therefore,

$$u = v(q) \Leftrightarrow d(u,q) = 1$$

Relationship of *d(u,q)* to *c(u,p)*:

$$d(u,q) \le \frac{q \bullet p}{c(u,p)}$$

proof: $c(u,p) \le q^* \bullet p,\qquad q^* = \dfrac{q}{d(u,q)} = q/OA$

$$\Rightarrow d(u,q)c(u,p) \le d(u,q)q^* \bullet p = q \bullet p$$
$$d(u,q) \le \frac{q \bullet p}{c(u,p)}$$

The inequality occurs because even though q^* is on *u*, it is not necessarily the cheapest bundle associated with *p*. However, if we choose a particular *p* such that the consumer buys a vector proportional to *q*, the above inequality becomes an equality. Hence, there exists at least one price vector *p* for which

$$d(u,q) = \frac{q \bullet p}{c(u,p)}$$

For any *p* this ratio is larger than *d(u,q)* (as we just showed) so that

$$d(u,q) = \min_p [p \bullet q : c(u,p) = 1] \tag{1}$$

The cost function can be defined as

$$c(u,p) = \min_q [p \bullet q : v(q) = u]$$

But $v(q) = u \Leftrightarrow d(u,q) = 1$. Therefore,

$$c(u,p) = \min_q [p \bullet q : d(u,q) = 1] \tag{2}$$

This relationship shows that the cost function and distance function are fundamentally duality relations.

Properties of distance function:

(a) Increasing in q and decreasing in u.

(b) Homogenous of degree one in q.

(c) Continuous and almost everywhere differentiable.

(d) Concave in q.

(e) Derivative property: $\dfrac{\partial d(u,q)}{\partial q_i} = a_i(u,q) = \dfrac{p_i}{x}$.

The $a_i(u,q)$ functions are homogenous of degree zero in quantities.

Inverse demand functions are frequently used when either prices do not exist or are artificially distorted. Thus, the prices can be viewed as shadow prices associated with u and ray q.

Economic interpretation of p_i: The amount of money individual on u and proportion q he is willing to pay for one more unit of q_i.

Uncompensated inverse demand functions:

$$\frac{p_i}{x} = a_i(u,q) = a_i[v(q),q] = b(q)$$

Antonelli matrix:

$$[a_{ij}] = \left[\frac{\partial^2 d}{\partial q_i \partial q_j}\right] = A$$

The Antonelli matrix is negative semi-definite, implying

$$a_{ij} = a_{ji} \qquad \forall i \neq j$$

$$a_{ii} \leq 0$$

Compensations are classified as

"q-complements" if $a_{ij} > 0$.

"q-substitutes" if $a_{ij} < 0$.

The Antonelli matrix A and the Slutsky matrix S are generalized inverses of one another.

We can also derive a Slutsky-type relationship by differentiating the equation for the uncompensated demand function

$$\frac{\partial a_i}{\partial u}\frac{\partial v}{\partial q_j} + \frac{\partial a_i}{\partial q_j} = \frac{\partial b_i}{\partial q_j}$$

But from the F.O.N.C. for utility maximization, $\frac{\partial v}{\partial q_j} = \lambda p_j$, so that

$$\frac{\partial b_i}{\partial q_j} = \frac{\partial a_i}{\partial u}\lambda p_j + \frac{\partial a_i}{\partial q_j}$$

Hotelling-Wold identity:

F.O.N.C. for utility maximization are:

$$\frac{\partial v(q)}{\partial q_i} = \lambda p_i \qquad\qquad i = 1,2,...,n$$

$$\sum p_k q_k = x$$

Multiply first set of equations by q_i and sum over all i:

$$\sum q_i \frac{\partial v}{\partial q_i} = \lambda \sum p_i q_i = \lambda x$$
$$\Rightarrow \lambda = \frac{\sum q_i \partial v / \partial q_i}{x}$$

Therefore,

$$\frac{P_i}{x} = b_i(q) = \frac{\partial v / \partial q_i}{\sum q_k (\partial v / \partial q_k)}$$

Scale aggregation condition:

Consider the more general form of the inverse demand functions

$$\frac{p_i}{x} = b_i(dq^*) \qquad \text{where} \quad d = q/q^*$$

The question we ask is how much will p_i change in response to proportional changes in the components of *q*?

$$\frac{\partial(p_i/x)}{\partial d} = \frac{\partial b_i}{\partial d} = \sum_j \frac{\partial b_i}{\partial q_j} q_j^* \tag{3}$$

The budget constraint can be written as

$$\sum (p_k/x) q_k = 1 \tag{4}$$

Differentiating with respect to q_j we obtain

$$\sum_k \frac{\partial(p_k/x)}{\partial q_j} q_k + (p_j/x) = 0$$

Multiply both sides by q_j:

$$\sum_k q_k \frac{\partial(p_k/x)}{\partial q_j} q_j + w_j = 0$$

Summing over all *j* and substituting from (3) we obtain

$$\sum_k q_k \sum_j q_j \frac{\partial(p_k/x)}{\partial q_j} + \sum_j w_j = 0$$
$$\Rightarrow$$

$$\sum_k q_k \frac{\partial b_k}{\partial d} = -1 \qquad (4)$$

which is called the <u>scale aggregation condition.</u>

Multiplying both sides of the Antonelli condition by q_j and summing over all j gives

$$\sum_j q_j \frac{\partial b_i}{\partial q_j} = \frac{\partial b_i}{\partial d} = \sum q_j \frac{\partial a_i}{\partial q_j} + \lambda \sum_j \frac{\partial a_i}{\partial u} p_j q_j$$
$$\Rightarrow \frac{\partial b_i}{\partial d} = \lambda \frac{\partial a_i}{\partial u} x$$

and

$$\lambda p_j \frac{\partial a_i}{\partial u} = \frac{\partial a_i}{\partial u} \lambda x \frac{p_j}{x} = \frac{\partial b_i}{\partial d} \frac{p_j}{x}$$

so that

$$\frac{\partial b_i}{\partial q_j} = \frac{\partial a_i}{\partial q_j} + \frac{\partial b_i}{\partial d} \frac{p_j}{x} \qquad (5)$$

Summary of restrictions on inverse demand functions:

$$\sum_k q_k \frac{\partial b_k}{\partial d} = -1 \qquad \text{(scale aggregation condition)} \qquad (6a)$$

$$\sum_k q_k \frac{\partial a_i}{\partial q_k} = 0, \quad i=1,2,\ldots,n \qquad \text{(homogeneity condition)} \qquad (6b)$$

$$\frac{\partial a_i}{\partial q_j} = \frac{\partial a_j}{\partial q_i}, \quad i,j=1,2,\ldots,n \qquad \text{(symmetry condition)} \tag{6c}$$

$$\frac{\partial a_i}{\partial q_i} < 0, \quad i=1,2,\ldots,n \qquad \text{(negativity condition)} \tag{6d}$$

$$\frac{\partial b_i}{\partial q_j} = \frac{\partial a_i}{\partial q_j} + \frac{\partial b_i}{\partial d}\frac{p_j}{x} \qquad \text{(Antonelli restriction)} \tag{6e}$$

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III. Empirical Analysis

Single-Equation Approach

Historically, single equation methodology used to estimate elasticities because: (a) interest was primarily in own-price and income elasticities for individual, specific commodities; (b) methodology from implementation of systems of demand functions approach not well developed until 1950s and 1960s; and (c) technological innovation of electronic computers and statistical procedures that lowered costs of estimation.

Stone's model:

$$\log q_i = \alpha_i + e_i \log x + \sum_{k=1}^{n} e_{ik} \log p_k \tag{1}$$

where q_i is per capita consumption of the *ith* commodity, x is per capita total expenditure, and p_k is the price of the *kth* commodity.

The model is expressed in per capita terms in order to neutralize the effects of changes in population. Behavior we are usually attempting to explain is that of "representative" consumer.

(Digression on aggregation problem) How can we justify this specification?

Let *h* denote a particular consumer so that $q_i^h = g_i^h(x^h, p)$. Per capita consumption is then defined as

$$\overline{q_i} = f_i(x^1,\ldots,x^H, p) = \frac{1}{H}\sum_h g_i^h(x^h, p)$$

Note that exact aggregation is possible only if we can write Marshallian demands in the special form:

$$\overline{q_i} = g_i(\overline{x}, p)$$

In general, no such function exists and the requirement that this function be consistent with utility maximization is more stringent than simply saying Marshallian demand functions exist. We require specifically that a dollar taken away from one person and given to another must leave the marginal propensities to consume unchanged. Therefore,

$$q_i = \alpha_i^h(p) + \beta_i(p)x^h$$

so that

$$\overline{q_i} = \alpha(p) + \beta(p)\overline{x}$$

It turns out that linearity of this type is both a necessary and sufficient condition for consistency with utility maximization. The cost function which generates this demand function must be what is called the *Gorman Polar Form:* $c^h(u^h, p) = a^h(p) + u^h b(p)$. Less restrictive forms of aggregation can be used, as indicated below in the context of the Almost Ideal Demand System.

The general restrictions in elasticity form can be represented as follows:

Adding-up: $\sum w_k e_k = 1$ (2a)

Homogeneity: $\sum e_{ik}^* = 0; \sum e_{ik} + e_i = 0$ (2b)

Symmetry: $w_i e_{ik}^* = w_k e_{ki}^*$ (2c)

Using the Slutsky equation $e_{ik}^* = e_{ik} + w_k e_i$ the symmetry condition can also be expressed as

$$e_{ik} = \frac{w_k}{w_i} e_{ki} + w_k (e_k - e_i)$$

Negativity: $e_{ii}^* < 0$ (2d)

Substituting the Slutsky equation for e_{ik} in (1) we have

$$\log q_i = \alpha_i + e_i \log(x/P) + \sum_{k=1}^{n} e_{ik}^* \log p_k \tag{3}$$

where $\log P = \sum_{k=1}^{n} w_k \log p_k$ (Geometric price index). The homogeneity restriction is now expressed as $\sum e_{ik}^* = 0$. This means we can express each price relative to some numeraire commodity, or using the same deflator as for X we can impose homogeneity on (3) to obtain

$$\log q_i = \alpha_i + e_i \log(x/P) + \sum_{k=1}^{n} e_{ik}^* \log(p_k / P) \tag{4}$$

Note that (a) income and prices are now in real rather than nominal values, and (b) the price elasticities are compensated rather than uncompensated elasticities.

Rewrite equation (4) as follows:

$$\log q_i = \alpha_i + e_i \log(x/P) + \sum_{k=1}^{n_1} e_{ik}^* \log(p_k / P) + \sum_{k=n_1+1}^{n} e_{ik}^* \log(p_k / P) \tag{5}$$

If $e_{ik}^* \cong 0$ for $k > n_1$ then we obtain the usual *Marshallian partial equilibrium demand specification*

$$\log q_i = \alpha_i + e_i \log(x/P) + \sum_{k=1}^{n_1} e_{ik}^* \log(p_k / P) \tag{6}$$

Intuitively, goods which are not close substitutes or complements would be expected to have smaller compensated elasticities. Thus, we can restrict our attention to just close set of commodities. This, indeed, is what is done in many applied studies.

Linear Expenditure System:

One problem with the Stone model is that there is no guarantee symmetry will be satisfied. In 1954, Richard Stone was successful in developing the first system of demand functions, consistent with the general restrictions of consumer behavior, which could be applied empirically. The Linear Expenditure System (LES) satisfies all the general restrictions of consumer behavior.

To derive the LES, start with the linear demand system:

$$q_i = \beta_i x + \beta_{ii} p_i + \sum_{k \neq i} \beta_{ik} p_k$$

Next impose the zero homogeneity restriction:

$$q_i = \beta_i (x / p_i) + \beta_{ii} + \sum_{k \neq i} \beta_{ik} (p_k / p_i)$$

or

$$p_i q_i = \beta_i x + \sum_{i=1}^{n} \beta_{ik} p_k \tag{1}$$

The adding-up property implies that $\sum \beta_i = 1, \sum_i \beta_{ik} = 0$. Also, Symmetry can be shown to imply $\beta_{ik} = \gamma_i \delta_{ik} - \beta_i \gamma_k$ (δ_{ik} =1 for i=k; 0 otherwise)

Substituting these restrictions into (1) gives the LES:

$$p_i q_i = p_i \gamma_i + \beta_i (x - \sum_k p_k \gamma_k) \tag{2}$$

where β_i >0.

The first term on the right-hand side of equation (2) is referred to as "committed expenditure" because it represents expenditures on a fixed bundle of consumption. The second term on the right-hand side of the equation is referred to as "supernumerary income" because it represents expenditures above the committed expenditures.

The utility function underlying the LES is the Stone-Geary utility function:

$$v(q) = \Pi (q_k - \gamma_k)^{\beta_k} \tag{3}$$

What does the cost function look like?

Note that from (2) there are only 2n parameters, 2n-1 of which are independent. How many restrictions are there generally for a demand system?

Adding-up: 1

Homogeneity: n

Symmetry: $(n^2-n)/2 = n(n-1)/2$

The total number of parameters to estimate is $n^2+n = n(n+1)$. Therefore, the total number of independent parameters to estimate (i.e., degrees of freedom) is

$$n(n+1)-[n+1+n(n-1)/2] = (n^2+n-2)/2.$$

This number is considerably more than 2n-1, the number of parameters to estimate for the LES. Why does the LES have so few parameters? The answer is that linearity is very restrictive—although it is less restrictive than assuming constant elasticities.

Limitations of the LES:

1. It does not admit inferior goods.
2. All goods must be net substitutes.
3. Additivity restriction on the direct utility function.
4. Absolute values of all uncompensated own-price elasticities are constrained to be less than their income elasticities.

A nice feature of the LES is that both the Cobb-Douglass (CD) and Constant Elasticity of demand functional forms (CE) are special cases of this model. Another feature of the model is that not only does condition (4) above hold, but also under certain circumstances it can be shown that all own-price elasticities are proportional to income elasticities where the constant of proportionality is the same. Why is this so?

Recall from our discussion of the decomposition of substitution effects into general and specific effects that

$$s_{ij} = \lambda U^{ij} - (\lambda / \lambda_x) \frac{\partial g_i}{\partial x} \frac{\partial g_j}{\partial x}$$

For $i \neq j$

$$s_{ij} = -(\lambda / \lambda_x) \frac{\partial g_i}{\partial x} \frac{\partial g_j}{\partial x}$$

Let $\phi = (\lambda / \lambda_x)/(1/x)$ be the "money flexibility". Then for $i \neq j$,

$$e_{ij}^{*} = s_{ij}\frac{p_j}{q_i} = -\phi x\frac{p_j}{q_i}\ \frac{\partial g_i}{\partial x}\frac{\partial g_j}{\partial x}$$

$$= -\phi\frac{p_j q_j}{x}\ \frac{\partial g_i}{\partial x}\frac{x}{q_i}\frac{\partial g_j}{\partial x}\frac{x}{q_j}$$

or

$$e_{ij}^{*} = -\phi w_j e_i e_j \qquad (4)$$

Note that in general, when $\lambda U^{ij} \neq 0$, that $e_{ij}^{*} = e_{ij}^{F} - \phi w_j e_i e_j$, where e_{ij}^{F} is the so-called *Frisch elasticity*, obtained by making an income compensation so that marginal rather than total utility is held constant.

Using the Slutsky equation expressed in elasticity form, $e_{ij}^{*} = e_{ij} + w_j e_i$, in equation (4) we obtain expressions for uncompensated cross-price elasticities:

$$e_{ij} = -e_i w_j (1+\phi e_j), \forall i \neq j \qquad (5)$$

For i=j, we can use the homogeneity condition

$$\sum_{j\neq i} e_{ij} + e_{ii} = -e_i$$

$$\sum_{j\neq i} e_{ij} = -e_i \sum_{j\neq i} w_j (1+\phi e_j)$$

$$= -e_i (\sum_{j\neq i} w_j + \phi \sum_{j\neq i} w_j e_j)$$

$$= -e_i [(1-w_i) + \phi(1-w_i e_i)]$$

Therefore, upon substituting (5) for the uncompensated elasticities, we obtain

$$e_{ii} = \phi e_i - e_i w_i (1 + \phi e_i) \qquad (6)$$

Note that when w_i is small $e_{ii} \approx \phi e_i$. That is, price elasticities are proportional to income elasticities. As we have shown above, with addivitivity of the direct utility function, we only require parameter estimates for n+1 parameters (n income elasticities plus the money flexibility) to obtain all n(n+1) price and income elasticities.

Testing the Theory

The LES and other functional forms (e.g., indirect addilog) satisfy the general restrictions exactly. Suppose we wish to test these restrictions. In principle, we can choose a functional form (flexible) with and without restrictions imposed and test for validity of the restrictions.

We must first recognize that there is a problem arising from the adding-up property. Consider again the set of Marshallian demand functions:

$$q_{it} = g_{it}(x_t, p_t) + \varepsilon_{it} \qquad i = 1,\ldots,n; t = 1,\ldots,T$$

Assume that $E(\varepsilon_t) = E(\varepsilon_{it},\ldots,\varepsilon_{nt})' = 0'$ and

$$E(\varepsilon_t \varepsilon_t') = E\begin{bmatrix} \varepsilon_{1t}^2 & \varepsilon_{1t}\varepsilon_{2t} & \varepsilon_{1t}\varepsilon_{nt} \\ \varepsilon_{2t}\varepsilon_{1t} & \varepsilon_{2t}^2 & \varepsilon_{2t}\varepsilon_{nt} \\ \varepsilon_{nt}\varepsilon_{1t} & \varepsilon_{nt}\varepsilon_{2t} & \varepsilon_{nt}^2 \end{bmatrix} = \Omega,$$

which is the time invariant covariance matrix of the vector of contemporaneous error terms.

Imposing the adding-up property implies

$$\sum_{i=1}^{n} p_{it}q_{it} = \sum_{i=1}^{n} p_{it}g_{it}(x_t, p_t) + \sum_{i=1}^{n} p_{it}\varepsilon_{it} = x_t$$

which implies

$$\sum_{i=1}^{n} p_{it}\varepsilon_{it} = 0$$

This means that the vector of disturbance terms is linearly dependent with at most rank = *n-1*. This also means that the contemporary variance-covariance matrix is a singular matrix

$$E(p_t'\varepsilon_t\varepsilon_t') = p_t'E(\varepsilon_t\varepsilon_t') = p_t'\Omega = 0'.$$

The problem occurs because of a redundancy in the system. For given income and prices, we have a system of *n+1*equations in *n* unknowns. Provided the disturbance terms are temporally uncorrelated, one can delete any one of the *n* relations and estimate the remaining *n-1* equations as a system without any loss of information.

Hypothesis Testing for Systems of Demand Functions:

For given *t* the log of the multivariate normal distribution is

$$log f_y(y_t) = -\frac{n-1}{2}\log(2\pi) - \frac{1}{2}log|\Omega_n| - \frac{1}{2}(y_t - g(x_t, p_t))'\Omega_n^{-1}(y_t - g(x_t, p_t))$$

where Ω_n is the variance-covariance matrix of the system of equations with the last row and last column deleted. When the error terms are *i.i.d.* the log likelihood function is

$$logL = \sum_{t=1}^{T} logf_y =$$

$$-\frac{T(n-1)}{2}\log(2\pi) - \frac{T}{2}log|\Omega_n| - \frac{1}{2}\sum_{t=1}^{T}(y_t - g(x_t, p_t))'\Omega_n^{-1}(y_t - g(x_t, p_t))$$

Minus twice the logarithm of the likelihood function is distributed chi-squared with number of restrictions equal to the difference in the number of parameters between the null and alternative hypotheses. The test statistic is therefore

$$-2logLR = -2(log\tilde{L} - log\check{L})$$

where $log\tilde{L}$ is the log likelihood value under the null hypothesis (restricted model) and $log\check{L}$ is the log likelihood value under the alternative hypothesis (unrestricted model).

Remark: Note that the computer output for SAS using PROC MODEL prints the log likelihood value directly so it is unnecessary to calculate it according to the formula above. The formula above is useful for analytical purposes showing explicitly what the actual statistic looks like.

Rotterdam Model:

Totally differentiate the Marshallian demand functions:

$$dq_i = \frac{\partial g_i}{\partial x}dx + \sum_k \frac{\partial g_i}{\partial p_k}dp_k$$

Note that $dq_i \equiv q_i d\log q_i, dx \equiv xd\log x, dp_k \equiv p_k d\log p_k$. Therefore, upon substituting into the total differentiable and dividing through both sides by q_i, we obtain

$$d\log q_i = e_i d\log x + \sum_k e_{ik} d\log p_k \tag{1}$$

Next substitute for the Slutsky equation to obtain:

$$d\log q_i = e_i (d\log x - \sum_k w_k d\log p_k) + \sum_k e^*_{ik} d\log p_k \tag{2}$$

These equations are simply the differential form of Stone's model; however this alone makes the model more general (why?). These equations do not satisfy symmetry but symmetry can be imposed by multiplying both sides of (2) by w_i:

$$w_i d\log q_i = w_i e_i (d\log x - \sum_k w_k d\log p_k) + \sum_k w_i e^*_{ik} d\log p_k \tag{3}$$

To turn (3) into an empirical model define the following terms:

$$\begin{aligned}
&\overline{w}_t = 0.5(w_t + w_{t-1}),\\
&b_i = p_i \partial g_i / \partial x,\\
&c_{ik} = w_i e^*_{ik},\\
&\Delta\log \overline{x}_t = \Delta\log x_t - \sum_k \overline{w}_t \Delta\log p_{kt}
\end{aligned}$$

Thus, the empirical counterpart to (3), called the *absolute price version* of the Rotterdam Model (RM) is

$$\overline{w}_{it} \Delta\log q_{it} = b_i \Delta\log \overline{x}_t + \sum_k c_{ik} \Delta\log p_{kt} + \varepsilon_{it} \tag{4}$$

where $\Delta \log q_{it} = \log q_{it} - \log q_{it-1}$, $\Delta \log p_{kt} = \log p_{kt} - \log p_{kt-1}$. The testable restrictions of this model are:

$\sum_k c_{ik} = 0$	$\forall i$	(homogeneity)
$c_{ik} = c_{ki}$	$\forall i, j (i \neq j)$	(symmetry)
$c_{ii} < 0$	$\forall i$	(negativity)

As can be seen from equation (3), the income and compensated price elasticities of the RM model can be obtained by dividing estimated coefficients by the budget share of the good in question. In turn, the Slutsky equation can be used to estimate uncompensated price elasticities.

With respect to actual empirical applications of the RM, the evidence on compatibility with general restrictions is mixed. Early work with this model indicated that with small number of commodities (less than or equal to nine) generally showed consistency with theory. Later work with more commodities suggested rejection of homogeneity but non-rejection of symmetry given homogeneity. Work by Theil and co-workers suggest that commonly used test statistics by failing to account for small sample sizes may overstate rejection.

Relative price version of RM:

Recall from comparative static results of decomposition of substitution effect

$$e_{ik}^{*} = \lambda U^{ik} \frac{p_k}{q_i} - (\lambda / \lambda_x) \frac{\partial g_i}{\partial x} \frac{\partial g_k}{\partial x} \frac{p_k}{q_i}$$

$$= \lambda U^{ik} \frac{p_k}{q_i} - \phi e_i e_k w_k$$

Let $v_{ik} = w_i \lambda U^{ik} \frac{p_k}{q_i}$. Therefore,

$$v_{ik} = w_i e_{ik}^* + \phi w_i e_i w_k e_k = c_{ik} + \phi b_i b_k \tag{5}$$

$$\sum_k v_{ik} = \phi b_i \tag{6}$$

Using (5) and (6) in (3) we obtain

$$\begin{aligned} w_i d \log q_i &= b_i d \log \bar{x} + \sum_k (v_{ik} - \phi b_i b_k) d \log p_k \\ &= b_i d \log \bar{x} + \sum (v_{ik} - \sum_j v_{ij} b_k) d \log p_k \\ &= b_i d \log \bar{x} + \sum_k v_{ik} d \log p_k - \sum_j v_{ij} (\sum b_k d \log p_k) \end{aligned}$$

or

$$w_i d \log q_i = b_i d \log \bar{x} + \sum_k v_{ik} (d \log p_k - \sum_j b_j d \log p_j) \tag{7}$$

The second term on the right-hand side of (7) is called the specific substitution component while the last term is referred to as the general substitution component. If preferences are want independent so that $v_{ik} = 0 \;\; \forall i \neq k$ then $v_{ii} = \phi b_i$. Therefore

$$w_i d \log q_i = b_i d \log \bar{x} + \phi b_i (d \log p_i - \sum_j b_j d \log p_j) \tag{8}$$

Equation (8) indicates that with want independence we can express demand for a given commodity as a function of real income and relative price of the own-good, where the denominator of the relative price is what is called the *marginal price index* as opposed to an average price index.

Flexible Functional Forms

Up until now we have demand equations for empirical work through one of two ways: (a) starting with a direct utility function (e.g., LES or Stone-Geary utility function), (b) starting with demand system and imposing and testing for general restrictions of consumer behavior (e.g., Rotterdam Model). A third approach is to use duality theory and start with either indirect utility function or cost function and give it enough parameters so that it is flexible enough to approximate an arbitrary representation of preferences.

Indirect Translog Model:

The model proposed by Christensen, Jorgenson and Lau (CJL) starts with the indirect logarithmic form of the indirect utility function:

$$log\varphi(x,p) = \ log\varphi\big((1/x)p\big) = \alpha_0 + \sum_{i=1}^{n} \alpha_i \log\left(\frac{p_i}{x}\right) + .5\sum_{j=1}^{n}\sum_{k=1}^{n} \beta_{jk} \log\left(\frac{p_j}{x}\right) \log\left(\frac{p_k}{x}\right)$$

The demand functions are obtained by application of Roy's identity:

$$w_i = \frac{p_i q_i}{x} = \frac{-\dfrac{\partial log\varphi(x,p)}{\partial \log p_i}}{\dfrac{\partial log\varphi(x,p)}{\partial \log x}}$$

We also know that because of the zero homogeneity property that

$$\frac{\partial log\varphi(x,p)}{\partial \log x} = -\sum \frac{\partial log\varphi(x,p)}{\partial \log p_k}$$

Substituting into the above expression gives

$$w_i = \frac{p_i q_i}{x} = \frac{\frac{\partial log\varphi(x,p)}{\partial \log p_i}}{\sum \frac{\partial log\varphi(x,p)}{\partial \log p_k}} = \frac{\frac{\partial log\varphi(x,p)}{\partial \log\left(\frac{p_i}{x}\right)}}{\sum \frac{\partial log\varphi(x,p)}{\partial \log\left(\frac{p_k}{x}\right)}}$$

Thus, the indirect translog is simply a ratio of price derivatives. The general form of the price derivative is

$$\frac{\partial log\varphi(x,p)}{\partial \log p_i} = \alpha_i + \sum \beta_{ki} \log\left(\frac{p_k}{x}\right)$$

Therefore, the CJL model is

$$w_i = \frac{\alpha_i + \sum \beta_{ki} \log\left(\frac{p_k}{x}\right)}{\sum \left(\alpha_j + \sum \beta_{kj} \log\left(\frac{p_k}{x}\right)\right)}$$

The model is homogenous of degree zero in the parameters (which can be seen by changing all the parameters by the same proportion). Thus, some normalization rule must be used. By convention, $\sum \alpha_j = -1$.

CJL show that we can impose both additivity and homogeneity constraints on the parameter estimates. If the indirect translog function is additive, $log\varphi(x,p) = F(\sum log\varphi^k\left(\frac{p_k}{x}\right))$, it must be the case that

$$\frac{\partial log\varphi(x,p)}{\partial \log\left(\frac{p_j}{x}\right)} = F' \frac{\partial log\varphi^j}{\partial \log\left(\frac{p_j}{x}\right)} = \alpha_j$$

$$\frac{\partial^2 log\varphi(x,p)}{\partial log\left(\frac{p_j}{x}\right) \partial log\left(\frac{p_k}{x}\right)} = F'' \frac{\partial log\varphi^j}{\partial \log\left(\frac{p_j}{x}\right)} \frac{\partial log\varphi^k}{\partial \log\left(\frac{p_k}{x}\right)} = \beta_{jk}$$

Thus, the function is *additive* if $\beta_{jk} = \theta\alpha_j\alpha_k,\ \theta = F''/(F')^2$. It will be *explicitly additive* if $\theta = 0$.

The indirect translog function will be homothetic if there exists a function *F* such that

$$log\varphi(x,p) = F(logH(x,p))$$

$$\frac{\partial logH(x,p)}{\partial \log x} = 1 = -\sum \frac{\partial logH(x,p)}{\partial \log p_k} = -\sum\left(\alpha_j + \sum \beta_{kj} \log\left(\frac{p_k}{x}\right)\right)$$

and $F' > 0$. With this form, it must be the case that $\frac{\partial log\varphi(x,p)}{\partial \log p_i} = F' \frac{\partial logH}{\partial \log p_i} = \alpha_i$. In addition,

$$\frac{\partial^2 log\varphi(x,p)}{\partial logp_j \partial logp_k} = F''\alpha_j\alpha_k + F' \frac{\partial^2 logH}{\partial \log p_j \partial \log p_k} = \beta_{jk}$$

Homotheticity implies (a) $1 = -\sum \frac{\partial logH(x,p)}{\partial \log p_k}$ and (b) $\sum_j \frac{\partial^2 logH}{\partial \log p_j \partial \log p_k} = 0$. Thus $\sum_j \beta_{jk} = F'' \sum_j \alpha_j \alpha_k = -F''\alpha_k = \sigma\alpha_k,\ \sigma = -F''$. A translog approximation to a homothetic function is not necessarily homothetic. A necessary and sufficient condition for homotheticity of the translog function is that it is homogenous. In that case $\sigma = 0$.

There are a total of (n-2)(n-1)/2 *additivity restrictions*, 1 *explicit additivity restriction*, n-1 *homotheticity restrictions*, and 1 *homogeneity restriction*.

The following connections between direct and indirect utility utility functions have been established:

(a) The direct utility function is homothetic if and only if the indirect utility function is homothetic.

(b) In general, an additive direct utility function does not correspond to an additive indirect utility function.

(c) If the direct utility function is additive and homothetic, the indirect utility function is additive and homothetic.

Remark: Uncompensated price elasticities and income elasticities can be obtained through differentiation of the functions in (2). As an aid to differentiation, note that

$$\frac{\partial \log w_i}{\partial \log p_k} = \delta_{ik} + e_{ik}; \quad \frac{\partial \log w_i}{\partial \log x} = -1 + e_i\,; \qquad \frac{\partial w_i}{\partial \log p_k} = \frac{\partial \log w_i}{\partial \log p_k} w_i\,;$$

$$\frac{\partial w_i}{\partial \log x} = \frac{\partial \log w_i}{\partial \log x} w_i\,.$$

Almost Ideal Demand System (AIDS):

Assume that the preferences are PIGLOG. This implies that the cost function has the form:

$$\log c(u, p) = a(p) + ub(p)$$

To impose curvature restrictions assume that the two price functions have the forms

$$a(p) = \alpha_0 + \sum_k \alpha_k \log p_k + 0.5 \sum_k \sum_l \gamma_{kl} \log p_k \log p_l \, ,$$

$$b(p) = \beta_0 \Pi p_k^{\beta_k} \, .$$

Homogeneity of the cost function is imposed by assuming

(a) $\sum_k \alpha_k = 1$,

(b) $\sum_k \gamma_{kl} = \sum_l \gamma_{lk} = 0$,

(c) $\sum_k \beta_k = 0$,

Symmetry is imposed through assuming $\gamma_{kl} = \gamma_{lk}$.

To derive demand functions from the PIGLOG preferences note that from Shephard's lemma:

$$\begin{aligned} w_i &= \frac{\partial \log c}{\partial \log p_i} = \frac{\partial a}{\partial \log p_i} + u \frac{\partial b}{\partial \log p_i} \\ &= \frac{\partial a}{\partial \log p_i} + ub \frac{\partial \log b}{\partial \log p_i} \end{aligned} \tag{1}$$

where $\frac{\partial a}{\partial \log p_i} = \alpha_i + \sum_k \gamma_{kl} \log p_k$; $\frac{\partial \log b}{\partial \log p_i} = \beta_i$.

The indirect utility function is obtained by inverting the cost function:

$$u = (\log c - a)/b = [\log x - a(p)]/b(p)$$

Substituting for (1) gives

$$w_i = \frac{\partial a}{\partial \log p_i} + \frac{[\log x - a(p)]}{b(p)} b(p) \frac{\partial \log b(p)}{\partial \log p_i} \qquad \text{or}$$

$$w_i = \alpha_i + \sum_k \gamma_{ki} \log p_k + \beta_i \log(x/P) \tag{2}$$

where

$$\log P \equiv a(p) = \alpha_0 + \sum_k \alpha_k \log p_k + 0.5 \sum_k \sum_l \gamma_{kl} \log p_k \log p_l \tag{3}$$

Properties of the AIDS model:

(a) Satisfies axioms of consumer choice exactly.

(b) Aggregates perfectly from a representative consumer.

(c) Engel curves are consistent with household data.

(d) Can be viewed as a first-order approximation to arbitrary demand system.

(e) Homogeneity and symmetry can be tested (i.e., $\sum_k \gamma_{ki} = 0; \gamma_{ki} = \gamma_{ik}$).

Remark: Muellbauer (1976) shows that given cost function for individual *h*, $w_{ih} = \partial \log c_h(u_h, p) / \partial \log p_i$, there exists a function $c(u_0, p)$ such that $\overline{w}_i = \partial \log c(u_0, p) / \partial \log p_i$, where $\overline{w}_i = p_i \sum_h q_{ih} / \sum x_h = \sum x_h w_{ih} / \sum x_h$ and $c_h(u_h, p)$ has the same properties as $c(u_0, p)$, whose income is $x_0 = c(u_0, p)$.

Elasticities of the AIDS model are obtained as follows:

$$\begin{aligned} e_{ik} &= \frac{\partial w_i}{\partial \log p_k} \frac{1}{w_i} - \delta_{ik} \\ &= \frac{\gamma_{ik} - \beta_i [w_k - \beta_k \log(x/P)]}{w_i} - \delta_{ik} \end{aligned}$$

$$e_i = \frac{\partial w_i}{\partial \log x} \frac{1}{w_i} + 1 = \frac{\beta_i}{w_i} + 1.$$

Relationships of AIDS to RM:

For the AIDS model:

$$w_i e_{ik}^* = \gamma_{ik} - \beta_i [w_k - \beta_k \log(x/P)] - \delta_{ik} w_i + w_k \beta_i + w_i w_k \qquad (1)$$

For the RM the variable on the left-hand side of (1) is a constant.

Remark: Mountain (1988) shows that the RM is at least as flexible as other so-called flexible functional forms (e.g., indirect translog, AIDS, generalized Leontief, miniflex Laurent). In other words, RM has errors which are at least of second order of magnitude.

Linear Approximate/Almost Ideal Demand System (LA/AIDS) Model:

AIDS model is often formulated in a different way for estimation

$$w_i = \alpha_i^* + \sum_k \gamma_{ki} \log p_k + \beta_i \log(x/P^*)$$

where $\log P^* = \sum w_k \log p_k$, i.e., the price index is approximated by the Stone index; and $\alpha_i^* = \alpha_i - \beta_i \alpha_0$. One reason for this model is that it makes estimation of the full AIDS model simpler because a nonlinear model has been converted to a model linear in parameters. It is also the case that this model avoids a potentially serious problem of multi-collinearity inherent in use of the price index $\log P$. However, there are problems associated with using the LA/AIDS model:

(1) The model is not desirable from the standpoint of restrictions of consumer behavior.

(2) It is unclear what the correct formulas for the elasticities should be; should they be derived from the LA/AIDS model or should they be based on the full AIDS model?

Moschini (1996) showed that the Stone index is not invariant to price scaling. He recommends using price indexes instead of actual prices for price variables in the model.

In an important paper, Asche and Wessells (1997) have reconciled the AIDS and LA/AIDS models by noting that at the point of approximation where all prices and income are normalized at unity, both the AIDS and the LA/AIDS have the same formulas for price and income elasticities.

There are other popular locally flexible demand models, too numerous to discuss here.

Miniflex Laurent:

In contrast to the popular locally flexible demand models discussed above, Barnett (1983) introduced a locally flexible functional form that has a large, but not global, regular region. Such functions typically have regular regions that include almost all the data points in the sample, and the regular regions increase as real total expenditure increases. There are other forms as well that satisfy this requirement: The quadratic AIDS and the general exponential form are two. The miniflex Laurent (ML) of Barnett can be specified as follows:

$$log\varphi(v) = \; log\varphi\left(\left(\frac{1}{x}\right)p\right) = \alpha_0 + \sum_{i=1}^{n} a_{ii}\, v_i + \sum_{i=1}^{n} 2\;\alpha_i\, \mathrm{v}_{\mathrm{i}}^{\frac{1}{2}} + \sum_{i\neq j}\sum_{i,j\in S} a_{ij}^2\, v_i^{\frac{1}{2}} v_j^{\frac{1}{2}}$$

$$-\sum_{i\neq j}\sum_{i,j\in S} b_{ij}^2\, v_i^{\frac{-1}{2}} v_j^{\frac{-1}{2}}$$

where $v_i = \frac{p_i}{x}$, and $S = \{(i,j): i \neq j; i,j = 1, \ldots, n\}$.

Gallant's Fourier Flexible Functional Form (FFF):

All the flexible functional forms discussed so far have been rationalized as second-order approximations of an arbitrary function, based on a second-order Taylor's series expansion. Gallant argues that Taylor's theorem has no known statistical properties. Moreover, we do not know where the point of approximation is in the sample or indeed whether the point of approximation lies in the sample (White 1980). Also, the errors in approximation by Taylor's theorem are not necessarily small. The advantage of the FFF is that it uniformly estimates both the function and its derivatives (which are used to calculate elasticities) up to some pre-specified order.

The FFF of order K (Gallant 1981) has the form:

$$g_k(x) = u_0 + b'x + 0.5x'Cx + \sum_{\alpha=1}^{A}\{u_{0\alpha} + 2\sum_{j=1}^{J}[u_{j\alpha}\cos(j\lambda k'_\alpha x) - v_{j\alpha}\sin(j\lambda k'_\alpha x)]\}$$

where x is an n-vector of logs of normalized prices p_i / y and the matrix C has the form:

$$C = -\sum_{\alpha=1}^{A} u_{0\alpha}\lambda^2 k_\alpha k'_\alpha .$$

The k_α's are multi-indexes which indicate the direction of the trigonometric expansion. The sequence of multi-indexes for *N=2* and *K=2* is:
$k_1 = (1,0)', k_2 = (0,1)', k_3 = (1,1)', k_4 = (1,-1)'$. The rules for determining the multi-indexes are discussed in Gallant (1981). The parameter λ is a scaling factor to ensure that the arguments of the cosine and sine functions (when *j=1*) are between *0* and 2π. The demand functions are obtained through application of Roy's identity using the differentiation rules: $\partial \sin(x)/\partial x_i = \cos(x), \partial \cos(x)/\partial x_i = -\sin(x).$

Nested PIGLOG (NEP) model (Piggott):

The idea behind this model is to generalize both the AIDS and indirect Translog models to include "pre-committed quantities:"

$$E(p,u) = p'c + E^*(p,u)$$

where the function $E(p,u)$ is the expenditure function and $E^*(p,u)$ is the expenditure function for supernumerary expenditures; $p'c$ represent pre-committed expenditures. For $E(p,u)$ to nest models that are consistent with PIGLOG preferences,

$$E(p,u) = p'c + \exp[a(p) + ub(p)]$$

The model that nests the AIDS, TL, and combines locally flexible models with the FFF is

$$E(p,u) = p'c + \exp[\delta + \alpha'\hat{P} + 0.5\hat{P}'\Gamma\hat{P} + \sum_{\alpha=1}^{A}\{u_\alpha \cos(\lambda k'_\alpha \hat{P}) - v_\alpha \sin(\lambda k'_\alpha \hat{P})\} + u\beta_0 \Pi p_k^{\beta_k}]/(\alpha'\iota + \hat{P}'\Gamma\iota)$$

The share equations have the form:

$$S = \frac{1}{M}\Phi + \left(\frac{M^*}{M}\right)[\alpha + \Gamma\Pi^* + \beta\{d(p)\ln M^* - \ln\tilde{P}] - 2\lambda\sum_{\alpha=1}^{A}\{u_\alpha \sin(\lambda k'_\alpha \hat{P}) + v_\alpha \cos(\lambda k'_\alpha \hat{P})\}k_d]/d(p)$$

where $d(p) = \alpha'\iota + \hat{P}'\Gamma\iota, \quad \Pi^* = \ln\left[\left(\frac{1}{M^*}\right)P\right],$

$\ln\tilde{P} = \delta + \alpha'\hat{P} + 0.5\hat{P}'\Gamma\hat{P} + 2\sum_{\alpha=1}^{A}\{u_\alpha \cos(\lambda k'_\alpha \hat{P}) - v_\alpha \sin(\lambda k'_\alpha \hat{P})\}, \quad \delta$ = N-vector of budget shares, and Φ = N-vector of pre-committed expenditurers ($\phi_i = p_i c_i$).

The empirical application is to food demand with three commodities: food at home, food away from home, and alcoholic beverages. Annual time series data from 1968-1999 are used. NEP model was estimated using iterated NSUR with alcoholic beverages equation eliminated. Prior to estimation, the following parameter values were specified: $\lambda = 6/(\max \ln \hat{p}_i)$, $\ln \hat{p}_i = \ln p_i - \min(\ln p_i) + \varepsilon$, $\forall i, \varepsilon = 0.00001$, $k_1 = (1,-1,0)', k_2 = (0,1,-1)', k_3 = (1,0,-1)'$.

In-sample tests indicated GAI model preferred model:

$$E(p,u) = p'c + \exp\left[\frac{\delta + \alpha'\hat{P} + 0.5\hat{P}'\Gamma\hat{P} + u\beta_0 \Pi_{k=1}^{N} p_k^{\beta_k}}{\alpha'\iota}\right].$$

Asymptotically Ideal Demand System (AIM):

One problem with the Fourier Flexible Functional Form is that it is hard to impose regularity conditions (i.e., concavity/convexity) on the form. Clearly, the functions estimated, while being able to consistently estimate elasticities, would not necessarily be considered *demand functions* if they failed to satisfy the basic restrictions of theory, i.e., negative semidefiniteness. Cognizant of this problem, Barnett et al have developed a model that satisfies the Sobolev space yet is able to globally approximate the true function and its derivatives within the set of concave functions. They derive its basic properties in the context of production theory (Barnett, Geweke, and Wolfe). Yue specified and estimated the AIM in the context of money demand for four different types of money (see article). The model includes the generalized Leontief as a special case. The model expands in a geometric fashion with both the number of goods and the number of terms in the expansion. The theory, like Gallant's Fourier Form, specifies that the number of the terms in the expansion is not fixed, but as the number of terms increases the approximation to the true function (and its derivatives) gets better.

While the model may be an improvement over the Fourier Model, it still has some drawbacks. First, all the coefficients must be non-negative for concavity to hold. Havener and Saha suggest imposing these constraints through replacing coefficients by

α_i^2 and differentiating with respect to the $\alpha_i's$. Second, the AIM has the same problem as the Fourier in that we do not know a priori how many terms to include in the expansion.

As we have discussed, the problem of selecting a functional form are inherently difficult because the true functional form is unknown. The same problem remains that any second-order parametric functional form may not produce consistent estimates of the elasticities.

<u>Endogeneity and Demand</u>

Suppose we are concerned that prices are not exogenous but are jointly determined by supply and demand. How does one test for the assumption that prices are exogenous? Thurman (1987) shows how to use the Wu-Hausman test to test for exogeneity of prices in demand for poultry. The two models he considers are as follows:

$$Q_t = bP_t^c + X_t'g + e_t \qquad (1)$$

$$P_t^c = dQ_t + X_t'h + u_t \qquad (2)$$

where Q_t = log of annual U.S. poultry meat consumption per capita, P_t^c = log of real (CPI-deflated) retail poultry meat price, and $X_t' = (P_t^b, P_t^p, I_t, 1)$ is a vector of predetermined variables (P_t^b = log of real retail beef meat price, P_t^p = log of real retail pork meat price, I_t = log of real per capita disposable personal income.).

Estimate the models by Instrumental Variables (IV) where the instrument vector is $W_t' = (X_t', Z_t')$, where $Z_t' = (P_t^f, P_t^l, P_t^e, C_t)$ (P_t^f = log of the price of feed, P_t^l = log of price of labor, P_t^e = log of the price of energy, and C_t = log of feed conversion.).

To test that P_t^c is predetermined in (10) and that Q_t is predetermined in (2) the relevant Wu-Hausman statistic is:

$$T = (\hat{b} - b^*)'[\hat{V}(q)]^{-1}(\hat{b} - b^*)$$

where $\hat{V}(q)$ is a consistent estimator of $n\,\mathrm{var}(q)$ under the null hypothesis H_0, $q = \hat{b} - b^*$, $\hat{b}$ is the estimator of b by OLS, and b^* is the IV estimator. Under H_0, $\tilde{T}$ is distributed chi-squared with the number of elements of q. Under H_0, the price variable in (1) will be exogenous or the quantity variable in (2) would be exogenous. Intuitively, the closer $\hat{b}$ is to b^* the less likely we are to reject the null hypothesis.

After controlling for structural shifts in demand, Thurman failed to reject the null hypothesis for exogenous price but reject the null hypothesis that quantity was predetermined. How do we interpret these findings? That is, why would we expect prices to be predetermined? One explanation could be the existence of constant returns to scale with horizontal factor supplies (i.e., existence of a constant-cost competitive industry). A second explanation could be an amorphous class of structures where prices are sluggish to respond to shifts in supply and demand. Thurman attempts to decide which of the two explanations is correct by testing to see if prices are correlated with lagged values of the demand disturbances. He finds that they are not. Therefore, the results are consistent with the first explanation given above.

Price Dependent Empirical Models

Sometimes it is more appropriate to use inverse demand models to estimate systems of demand functions. This is the case for commodities whose supply curves are very nearly or perfectly inelastic. One commodity that has been studied quite extensively is fish.

Inverse Almost Ideal Demand System (IAIDS) (Eales and Unnevehr 1993):

Let the distance function be defined as

$$\log D(u,q) = (1-u)\log a(q) + u\log b(q)$$
$$\log a(q) = \alpha_0 + \sum_j \alpha_j \log q_j + 0.5\sum_i \sum_j \gamma_{ij} \log q_i \log q_j$$
$$\log b(q) = \beta_0 \Pi q_j^{-\beta_j} + \log a(q)$$

Recall that the derivative of the distance function gives compensated demand functions. Therefore,

$$\frac{\partial \log D(u,q)}{\partial \log q_i} = w_i = \alpha_i + \sum_j \gamma_{ij} \log q_j - \beta_j \ \beta_0 \Pi q_j^{-\beta_j} u$$

Inversion of the distance function at the optimum yields the direct utility function

$$v(q) = \frac{-\log a(q)}{[\log b(q) - \log a(q)]}$$

Substituting into the above compensated inverse demand function gives

$$w_i = \alpha_i + \sum_j \gamma_{ij} \log q_j + \beta_j \log Q$$
$$\log Q \equiv \log a(q)$$

The restrictions of this model are as follows:

$\sum \alpha_i = 1, \sum \sum \gamma_{ij} = 0, \sum \beta_i = 0$ (adding-up)

$\sum_j \gamma_{ij} = 0$ (homogeneity)

$\gamma_{ij} = \gamma_{ji}$ (symmetry)

The linear approximate IADS model (LA/IADS) is analogous to the LA/AIDS model:

$$w_i = \alpha_i + \sum_j \gamma_{ij} \log q_j + \beta_i \log Q^*$$
$$\log Q^* = \sum_j w_j \log q_j$$

The inverse RM, due to Barten and Betandorf, is obtained in analogous manner to the RM model. Let $\pi_i = p_i / x = b_i(q)$. Totally differentiate this system and convert to logs to obtain

$$d \log \pi_i = \sum f_{ij} d \log q_j$$

Substituting the Slutsky-type restriction $f_{ij} = f_{ij}^* + w_j f_i$ yields the expression

$$d \log \pi_i = \sum f_{ij}^* d \log q_j + f_i d \log Q$$
$$d \log Q = \sum w_j d \log q_j$$

Multiplying both sides by the expenditure shares we obtain:

$$w_i d \log \pi_i = \sum \alpha_{ij} d \log q_j + \alpha_i d \log Q \qquad (1)$$

In log differential form the LA/IADS model is

$$dw_i = \sum_j \beta_{ij} d \log q_j + \beta_i d \log Q \qquad (2)$$

The two models are related as follows (Eales et al):

$$\begin{aligned} \alpha_i &= \beta_i - w_i \\ \alpha_{ij} &= \beta_{ij} - w_i \delta_{ij} + w_i w_j \end{aligned} \qquad \text{(Antonelli effects)}$$

Two other models, the inverse CBS (Central Bureau of Statistics) and inverse NBR (National Bureau of Research) are combinations of (1) and (2):

$$\begin{aligned} & w_i d \log(p_i / P) = \beta_i d \log Q + \sum_j \alpha_{ij} d \log q_j \\ & d \log P = \sum_i w_i d \log p_i \end{aligned} \tag{3}$$

$$dw_i - w_i d \log Q = \alpha_i d \log Q + \sum_j \beta_{ij} d \log q_j \tag{4}$$

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IV. Separability and Commodity Aggregation

Recall that the system of demand functions has $0.5(n^2 + n - 2)$ total number of parameters to estimate. Even for a moderately sized system, this becomes a rather large set of parameters to estimate. So some aggregation of prices seems appropriate.

Hicks Composite Commodity Theorem (Deaton and Muellbauer, chapt.5):
If a group of prices move proportional to one another, then the collection of goods in a group can be treated as a single good.

Suppose $c = c(u, p_1, p_2, p_3)$. Let $p_2 = \theta p_2^0, p_3 = \theta p_3^0 \Rightarrow \theta = \dfrac{p_2}{p_2^0} = \dfrac{p_3}{p^0}$. That is, assume constant price proportionality. Also let $Q = p_2^0 q_2 + p_3^0 q_3$ be constant dollar expenditures and define $c^*(u, p_1, \theta) = c(u, p_1, \theta p_2^0, \theta p_3^0)$. By Shephard's lemma,

$$\frac{\partial c^*}{\partial \theta} = \frac{\partial c}{\partial p_2} p_2^0 + \frac{\partial c}{\partial p_3} p_3^0 = q_2 p_2^0 + q_3 p_3^0 = Q$$

so that θ is the price of the composite good, whose quantity is Q.

Price proportionality is quite limited so there is obviously interest in a more general approach to commodity aggregation that focuses on restrictions on preferences. Separability—restrictions on consumer preferences—is a common way to approach this problem. There are three main concepts of separability:

1. Weak separability: the utility function $v(q)$ is weakly separable with respect to the partition $\{n_1, n_2, \ldots, n_N\}$ if and only if

$$v(q) = F[v_1(q_1), \ldots, v_N(q_N)])$$

where $q_i = (q_1,...q_{n_i})'$ is an $n_i x1$ vector of commodities for the *ith* subgroup. Weak separability also implies that

$$\frac{\partial[\partial v/\partial q_i)/\partial v/\partial q_k)]}{\partial q_j} = 0 \qquad \text{for } i,k \in G,\ j \in H, G \neq H$$

and

$$s_{ik} = \mu_{GH} \frac{\partial g_i}{\partial x}\frac{\partial g_k}{\partial x} \qquad \text{for } i \in G, k \in H, G \neq H.$$

2. Strong separability: $v(q)$ is strongly separable with respect to the partition $\{n_1, n_2,..., n_N\}$ if and only if

$$v(q) = F[v_1(q_1) + ... + v_N(q_N)]$$

$$\frac{\partial[(\partial v/\partial q_i)/(\partial v/\partial q_k)]}{\partial q_j} = 0 \qquad \text{for } i,k \in G,\ j \in H, G \neq H,$$

$$s_{ik} = \mu \frac{\partial g_i}{\partial x}\frac{\partial g_k}{\partial x} \qquad \text{for } i \in G, k \in H, G \neq H.$$

3. Implicit separability: $u = v(q)$ is implicitly separable if and only if the cost function can be written in the form

$$c(u, p) = c[u, c_1(u, p_1),..., c_N(u, p_N)]$$

where $p_i = (p_1,..., p_{n_i})'$ is the price vector associated with the *ith* subgroup.

Implications of weak separability:

Weak separability allows us to write demand functions entirely in terms of expenditure on the group in question and prices of commodities within that group, i.e.,

$$q_i = g_i^G(x_G, p_G) \qquad \text{for } i \in G$$

$$x_G = \sum_{k \in G} p_k q_k, \qquad p_G = (p_1, \ldots, p_{n_G})'.$$

Proof:

Assume $v(q) = F\{v_1(\bar{q}_1), \ldots, v_{G-1}(\bar{q}_{G-1}), v_G(q_G), v_{G+1}(\bar{q}_{G+1}), \ldots, v_N(q_N)]$ where $\bar{q}_i$ Denotes the pre-allocated or predetermined amount maximizing utility subject to the budget constraint yields

$$\frac{\partial v / \partial q_i}{\partial v / \partial q_k} = \frac{(\partial F / \partial V_G)(\partial V_G / \partial q_i)}{(\partial F / \partial V_G)(\partial V_G / \partial q_k)} = \frac{\partial V_G / \partial q_i}{\partial V_G / \partial q_k} = \frac{p_i}{p_k} \tag{1}$$

$$\sum_{i \in G} p_i q_i + \sum_{j \in H} p_j \bar{q}_j = x, G \neq H. \tag{2}$$

Since the $n_G - 1$ equations in (1) are independent of q_j for $j \in H, H \neq G$, it follows that the conditional demand functions can be written as

$$q_i = \bar{g}_i(x - \sum_{j \in H} p_j \bar{q}_j, p_G) \tag{3}$$

Suppose now that $\bar{q}_j = g_j(x, p)$. Then

$$q_i = \bar{g}_i(x - \sum_{j \in H} p_j g_j(x, p), p_G) = g_i^G(x_G, p_G) \quad (4)$$

This means that, provided the commodities appearing in the *Gth* group are weakly separable from all other commodities in the consumer's choice set, the conditional demand functions with income $x_G = x - \sum_{j \in G} p_j q_j$ and prices $p_G = (p_1, ..., p_{n_G})'$ can be treated in every respect as a system of Marshallian demand functions.

Two-stage budgeting:

Because of the degrees of freedom problem (i.e., too many parameters to estimate simultaneously), economists have proposed that the decision problem of the consumer be viewed as a utility tree where the large branches of the tree would correspond to different basic "wants" such as food, housing, clothing, recreation, etc., and the smaller branches of each main branch would correspond to the individual items purchased within those categories (e.g., under food we might have meats, fruits and vegetables, cereal and bakery products, dairy products, etc.). One might even envision a multi-stage process where each small branch could be divided into smaller branches to correspond to items within each one of the smaller categories. (e.g., under meats we might have beef, pork, poultry, fish, etc.). With such a preference structure, the consumer can then be viewed as first allocating her budget across the large categories (food, housing, clothing, recreation, etc.), and then in subsequent stages allocating the predetermined expenditure for that large category among the items within that group (e.g., for food, allocation of food expenditures among the main food categories). The question from a conceptual point of view is, under what conditions will the solutions to this multi-stage problem be consistent with the single-stage consumer optimization problem?

For simplicity, consider the two-stage maximization problem:

First Stage

Maximize $\quad u = v(Q^1,...,Q^N)$

Subject to: $\quad \sum_{G=1}^{N} P^G Q^G = x$

Second Stage

Maximize $\quad Q^G = Q^G(q_1,...q_{n_G})$

Subject to: $\quad \sum_{i \in G} p_i q_i = x_G$

The solution to this two stage problem should be identical to the solution to the single-stage problem, i.e.,

$$q_i = g_i(x, p) = g_i^G(x_G, p_G)$$

We have already shown that the second stage problem will yield the same solution, provided that weak separability holds and x_G represents optimal expenditure on the *Gth* group. But $P^G Q^G = x_G = x_G(P^1,...,P^N, x)$, so the question becomes: under what conditions will $x_G(P^1,...,P^N, x) = x_G(x, p)$? This is the *consistency requirement* which can alternatively be stated as follows:

$$\frac{\partial x_G / \partial p_j}{\partial x_G / \partial p_k} = \frac{\partial P^H / \partial p_j}{\partial P^H / \partial p_k} \qquad \forall j, k, G, H$$

When will this condition hold? In general, when either weak separability plus either homogenous separability or strong separabililty holds. The following theorems from Green specify the requirements.

<u>*Theorem 1 (Gorman 1959):*</u>

The utility function, $u = v(q_1,...q_n)$, may be written in the form

$$u = \psi(\psi_1, ..., \psi_N)$$

where $\forall G = 1,..., N \quad (n_1 + n_2 + ... + n_N = n)$

$$\psi_G = \psi_G[x_G(x, P^1, ..., P^N), p_G] \qquad \text{and}$$

$$x = \sum_{G=1}^{N} x_G$$

only if the weak separability condition holds and

either (i) there are only two groups:

or (ii) each group quantity index can be written in direct form as a homogeneous function of the quantities in the group;

or (iii) the quantity indices for all groups but one (say the first) are homogeneous, and *u* may be written as $u = \psi[\psi_1, H(\psi_2, ..., \psi_N)]$;

or (iv) the quantity indices for all groups beyond the *Gth* are homogeneous, and *u* may be written in the following additive form:

$$u = \psi[\sum_{r=1}^{G} G_r(\psi_r) + H(\psi_{G+1}, ..., \psi_N)].$$

<u>*Theorem 2 (Green, p.25):*</u>

The necessary and sufficient conditions for both (a) the consistency of the two-stage maximization procedure, and (b) the existence for each group of a quantity index Q^G and a price index P^G such that group expenditures $x_G = P^G Q^G, G = 1,..., N,$ are that each quantity index be a function homogenous of degree one in its elementary commodities or inputs.

Let's now look closely at the two important cases of homogenous separability and strong separability, since these are the two cases most used in applied work.

Homogeneous separability:

Note that weak separability implies that the indirect utility function for the *Gth* group can be written as $u_G = \psi_G[x_G(x, p), p_G]$ so that the first-stage problem can be restated as follows:

$$\text{Maximize} \qquad u = F(u_1, ..., u_N)$$

$$\text{Subject to:} \qquad \sum_G c_G(u_G, p_G) = x$$

Now homogenous separability implies that $c_G(u_G, p_G) = u_G b_G(p_G)$ so that F.O.N.C. for utility maximization can be written as

$$\frac{\partial F / \partial u_G}{\partial F / \partial u_H} = \frac{P_G}{P_H} = \frac{b_G(p_G)}{b_H(p_H)}$$
$$\sum_G u_G b_G(p_G) = x$$

The solutions can therefore be written as

$$x_G = x_G(x, P^1, ..., P^N).$$

Strong separability:

Strong separability implies the cost function for each group can be written as follows:

$$c_G(u_G, p_G) = a_G(p_G) + u_G b_G(p_G)$$

which you should recognize as the Gorman Polar Form. To see that this is so, let

$$u_G = F_G(v_G) - a_G / b_G$$

When the utility is in the GPF we have

$$F[v(q)] = \sum_G u_G = \sum_G F_G(v_G) - \sum_G a_G / b_G \; .$$

For given prices

$$\text{Maximize} \qquad \sum_G u_G$$

$$\text{Subject to} \qquad \sum_G c_G(u_G, p_G) = x$$

is equivalent to the problem

$$\text{Maximize} \qquad \sum_G F_G(v_G)$$

$$\text{Subject to} \qquad \sum_G (a_G + u_G b_G) = \sum_G (a_G + F_G b_G - \frac{a_G}{b_G} b_G) = \sum_G F_G b_G = x.$$

The solution to this problem satisfies the consistency requirement when $b_G = P^G$, etc.

PIGLOG model as GPF:

Lewbel (1989) shows that PIGLOG model can be written as GPF and therefore satisfies the requirement for existence of price indexes. Recall that $u_G = (\log X_G - \log a_G)/b_G$ Taking logs of both sides then gives

$$\log u_G = \log(\log x_G - \log a_G) - \log b_G = \log(\log(x_G / a_G) - \log b_G$$

which is precisely the GPF when we define $u_G = x_G / a_G$.

System-wide Analysis of Demand

How can we estimate demand for a group of commodities in two or more steps in such a way that our results are consistent with those obtained through single-stage maximization? Specifically, how can we estimate demand elasticities for a group of commodities (e.g., meats—beef and veal, pork, poultry, fish) by economizing on degrees of freedom through aggregating prices of goods outside the group in question such that

$$q_i = g_i^G[x_G(x, P^1, ..., P^N), p_G] = \bar{g}_i(x, P^1, ..., P^N, p_G) = g_i(x, p)$$

According to theorem 1, the least restrictive way of doing this is to assume strong separability or want independence between groups of commodities.

Let's consider the log differential approach of Theil, as shown by Clements and Johnson, to derive the relevant restrictions. Recall that the relative price version of the RM is

$$w_i d\log q_i = b_i d\log \bar{x} + \sum_k v_{ik}(d\log p_k - d\log P') \qquad (1)$$

where $d\log P' = \sum_{j=1}^{n} b_j d\log p_j$ is the marginal price index. When preferences are block independent, $v_{ik} = 0$ for $i \in G, k \in H, G \neq H$. This implies that $\sum_{k=1}^{n} v_{ik} = \sum_{k \in G} v_{ik} = \phi b_i$ and equation (1) can be rewritten as

$$w_i d\log q_i = b_i d\log \bar{x} + \sum_{k \in G} v_{ik} d\log(p_k / P') \qquad (2)$$

so that the only deflated prices appearing in (2) are goods belonging to group G. Let $w_G = \sum_{i \in G} w_i$ and $b_G = \sum_{i \in G} b_i$ denote average and marginal price budget shares of group G, respectively. Also, define $d\log\bar{x}_G = \sum_{i\in G}(w_i / w_G)d\log q_i$ and

$d\log P'_G = \sum_{i\in G}(b_i / b_G)d\log p_i$. Summing over the left-hand side of (2) we obtain

$$\sum_{i\in G} w_i d\log q_i = w_G d\log\bar{x}_G = \sum_{i\in G} b_i d\log\bar{x} + \sum_{i\in G}\sum_{k\in G} v_{ik}(d\log p_k - d\log P')$$
$$= b_G d\log\bar{x} + \sum_{k\in G}\sum_{i\in G} v_{ki}(d\log p_k - d\log P')$$

because of symmetry of v_{ik}. But $\sum_{i\in G} v_{ki} = \phi b_k$ so that upon substituting above we have

$$w_G d\log\bar{x}_G = b_G d\log\bar{x} + \phi b_G d\log(P'_G / P') \qquad (3)$$

which is called the *Composite demand equation* for good G.

Conditional demand functions for individual commodities within group G can be obtained through combining equations (2) and (3), or more simply by noting that the absolute price version of the RM can be written as

$$w_i d\log q_i = b'_i(w_G d\log\bar{x}_G) + \sum_{k\in G} c^G_{ik} d\log p_k \qquad (4)$$

where $b'_i = b_i / b_G, c^G_{ik} = w_i e^{G*}_{ik}, \sum_i b'_i = 1, \sum_{k\in G} c^G_{ik} = 0, c^G_{ik} = c^G_{ki}$.

Unconditional demand responses are obtained by combining (3) and (4):

$$w_i d\log q_i = b'_i b_G d\log\bar{x} + \sum_{k\in G}[c^G_{ik} + \phi(1-b_G)b_G b'_i b'_k]d\log p_k \qquad (5)$$
$$- b'_i\phi b_G \sum_{k\notin G} b_k d\log p_k$$

From equation (5) we infer that

$$e_i = \frac{b_i' b_G}{w_i} \tag{6}$$

$$e_{ik}^* = \frac{c_{ik}^G + \phi(1-b_G) b_G b_i' b_k'}{w_i} = e_{ik}^{G*} + \frac{\phi(1-w_G e_G) w_G e_G (w_i / w_G) e_i^G (w_k / w_G) e_k^G}{w_i} =$$
$$e_{ik}^{G*} + e_G \phi(1-w_G e_G) e_i^G w_k^G e_k^G$$
$$i,k \in G \qquad (7)$$

$$e_{ik}^* = \frac{-\phi b_i' b_G b_k}{w_i} \quad , \qquad i \in g, k \notin G \tag{8}$$

For empirical implementation, estimate (3) and (4) separately and synthesize unconditional demand responses using (6)-(8). To implement empirically we need to know weights on Frisch (i.e., marginal) price index. Clements and Johnson propose the approximation

$$d \log P' = b_G \sum_{j \in G} b_j' d \log p_j + (1-b_G) \sum_{k \notin G} \frac{b_k}{(1-b_G)} d \log p_k$$

where $\sum_{k \notin G} \frac{b_k}{(1-b_G)} d \log p_k \approx d \log P_0$,

$d \log P_0 = [1/(1-w_G)](d \log CPI - \sum_{j \in G} w_j d \log p_j)$.

<u>"Approximate" Multi-Stage Demand Systems (Edgerton, Carpentier and Guyomard)</u>

One problematic aspect of the requirement for consistency of multi-stage budgeting with single-stage optimization is that the price and quantity indexes will not be independent of income and/or utility with only the restriction of weak separability. Additional restrictions such as homogenous separability or strong separability are required for consistency. Another approach to the problem, though, is to consider approximations to the true price and quantity indexes based on observed data and ask how close these approximations are. As discussed by Edgerton, justification for approximate price and quantity indexes is as follows:

(i) the TCOL (True cost of living index) is

(u_r = reference utility; π_G = base year prices)

$$P^G = \frac{c_G(u_r, p_G)}{c_G(u_r, \pi_G)}$$

TCOL only completely invariant to u_r when preferences are homothetic. If P^G only varies slightly with u_r then approximate justification exists.

(ii) Some limited empirical evidence exists that TCOL does indeed vary only slightly with u_r. More commonly, however, price indexes are collinear. If true, then TCOL can be approximated by either the Paasche or Laspeyres index.

(iii) Price indexes of the form $\sum w_k p_k$ will be highly correlated with each other if the number of elementary commodities is large.

(iv) D&M (p. 174) used a Taylor's series expansion of the cost function around (u_r, π_G) to show that a first-order approximation to TCOL is

$$P^G = \frac{\bar{q}'_G p_G}{\bar{q}'_G \pi_G} \qquad \text{(Laspeyres price index)}$$

where $\bar{q}'_G$ is the reference quantity bundle given by Hicksian demand function $h(u_G, \pi_G)$.

Implications for calculating elasticities:

$$e_i = \frac{\partial \log g_i^G}{\partial \log x_G} \frac{\partial \log x_G}{\partial \log x}$$

$\log x_G = \log g_G + \log P^G$, because $q_G = g_G(P^1 ..., P^N, x)$

$$e_i = e_i^G \left(\frac{\partial \log g_G}{\partial \log x} + \frac{\partial \log P^G}{\partial \log x} \right) = e_i^G \left(e_G + \frac{\partial \log P^G}{\partial \log x} \right) \tag{5}$$

Assume P^G is approximately independent of the level of expenditure, and thus that

$$e_i = e_i^G e_G \tag{6}$$

Implications for price elasticities (Carpentier and Guyomard):

$$s_{ij} = \left.\frac{\partial g_i^G}{\partial p_j}\right|_U + \left.\frac{\partial q_i}{\partial x_G} \frac{\partial x_G}{\partial p_j}\right|_U \qquad i, j \in G$$

$$= \qquad \left.\frac{\partial q_i}{\partial x_G} \frac{\partial x_G}{\partial p_j}\right|_U \qquad i \in G, j \in H, G \neq H$$

But $s_{ji} = \left.\frac{\partial q_j}{\partial x_H} \frac{\partial x_H}{\partial p_i}\right|_U$ between groups. By symmetry, $s_{ij} = s_{ji} \Rightarrow$

$$\left.\frac{\partial q_i}{\partial x_G} \frac{\partial x_G}{\partial p_j}\right|_U = \left.\frac{\partial q_j}{\partial x_H} \frac{\partial x_H}{\partial p_i}\right|_U \qquad \Rightarrow \qquad \frac{\left.\frac{\partial x_G}{\partial p_j}\right|_U}{\frac{\partial q_j}{\partial x_H}} = \frac{\left.\frac{\partial x_H}{\partial p_i}\right|_U}{\frac{\partial q_i}{\partial x_G}}$$

Because the left-hand side of the expression on the right only involves j and right-hand side only involves I, we have that

$$\left.\frac{\partial x_G}{\partial p_j}\right|_U = \lambda_{GH}\frac{\partial q_j}{\partial x_H}$$

Therefore,

$$s_{ij} = \left.\frac{\partial q_i}{\partial x_G}\frac{\partial x_G}{\partial p_j}\right|_U = \lambda_{GH}\frac{\partial q_i}{\partial x_G}\frac{\partial q_j}{\partial x_H}$$

Note also that $\sum_{j\in H} p_j \left.\frac{\partial x_G}{\partial p_j}\right|_U = \lambda_{GH}\sum_{j\in H} p_j \frac{\partial q_j}{\partial x_H} = \lambda_{GH}$

Symmetry implies $\lambda_{GH} = \lambda_{HG}$, and homogeneity implies $\sum_G \lambda_{GH} = \sum_H \lambda_{HG} = 0$

Within group:

$$s_{ij} = \left.\frac{\partial g_i^G}{\partial p_j}\right|_U + \left.\frac{\partial q_i}{\partial x_G}\frac{\partial x_G}{\partial p_j}\right|_U = s_{ij}^G + \lambda_{GG}\frac{\partial q_i}{\partial x_G}\frac{\partial q_j}{\partial x_G}$$

But $\lambda_{GH} = \sum_{j\in H} p_j \left.\frac{\partial x_G}{\partial p_j}\right|_U$ Also $\left.\frac{\partial x_G}{\partial p_j}\right|_U = P^G \left.\frac{\partial Q^G}{\partial p_j}\right|_U = P^G \left.\frac{\partial Q^G}{\partial P^H}\right|_U \frac{\partial P^G}{\partial p_j} = P^G \left.\frac{\partial Q^G}{\partial P^H}\right|_U \frac{h_j}{\bar{c}_H}$

(because j comes from group H)

which implies that $\lambda_{GH} = \sum_{j\in H} p_j \left.\frac{\partial x_G}{\partial p_j}\right|_U = \sum_{j\in H}\frac{p_j h_j}{\bar{c}_H} P^G \frac{\partial H^G}{\partial P^H} = \frac{c_H}{\bar{c}_H} P^G \frac{\partial H^G}{\partial P^H} = P^G P^H \frac{\partial H^G}{\partial P^H}$

Therefore,

$$s_{ij} = \delta_{GH} s_{ij}^G + P^G P^H \frac{\partial H^G}{\partial P^H}\frac{\partial q_i}{\partial x_G}\frac{\partial q_j}{\partial x_G}$$

Converting to elasticities we obtain

$$e_{ij}^{*} = \delta_{GH} e_{ij}^{*G} + w_j^H e_{GH}^{*} e_i^G e_j^H$$

Also

$$e_{ij} = e_{ij}^{*} - w_j e_i$$

and

$$e_i = e_G e_i^G$$

<u>Testing Weak Separability</u>

Recall the condition that weak separability imposes on the substitution effects:

$$s_{ik} = \mu_{GH} \frac{\partial g_i}{\partial x} \frac{\partial g_k}{\partial x} \tag{1}$$

This is a necessary and sufficient condition for weak separability and it, therefore, summarizes all the relevant restrictions of the structure

$$v(q) = V[v_1(q_1),...,v_N(q_N)]$$

Take any two goods from group G (say i,j) and any two goods from group H (say k,m). Then it follows that for G $\neq$ H

$$\frac{s_{ik}}{s_{jm}} = \frac{(\partial g_i / \partial x)(\partial g_k / \partial x)}{(\partial g_j / \partial x)(\partial g_m / \partial x)} \tag{2}$$

Recall that $e^*_{ik} = s_{ik}(p_k / q_i), e_i = (\partial g_i / \partial x)(x / q_i)$, etc., so that

$$\frac{e^*_{ik}(q_i / p_k)}{e^*_{jm}(q_j / p_m)} = \frac{e_i(q_i / x)e_k(q_k / x)}{e_j(q_j / x)e_m(q_m / x)} \qquad \text{or}$$

$$\frac{e^*_{ik} / p_k q_k}{e^*_{jm} / p_m q_m} = \frac{e_i e_k}{e_j e_m} \tag{3}$$

But $e^*_{ik} / w_k = \sigma_{ik}$, etc., so that (3) can be expressed as

$$\frac{\sigma_{ik}}{\sigma_{jm}} = \frac{e_i e_k}{e_j e_m} \tag{4}$$

$\forall i, j \in G; k, m \in H; G \neq H.$

Equation (4) defines a set of restrictions that can be maintained in any of the commonly used demand systems, or subjected to statistical tests. Note that if preferences are homothetic $e_i = e_j, e_k = e_m$, implying $\sigma_{ik} = \sigma_{jm}$. In general, there are a total of 0.5[$n(n-1) - \sum_{G=1}^{N} n_G(n_G - 1) - N(N-1)]$ non-redundant restrictions implied by (4).

Implications of weak separability test for particular functional forms:

(i) AIDS

$$\begin{aligned} \sigma_{ik} &= \frac{\gamma_{ik}}{w_i w_k} + \frac{\beta_i}{w_i}\frac{\beta_k}{w_k}\log(x / P) + 1 \\ e_i &= 1 + \frac{\beta_i}{w_i} \end{aligned} \qquad (i \neq k)$$

Restrictions from (4) will not hold globally unless $\beta_i = \beta_k = \beta_j = \beta_m = 0$ and $\gamma_{ik} = \gamma_{jm} = 0$. These parametric restrictions imply homothetic preferences and added restriction that $e_i = 1$.

(ii) RM

Recall that $e^*_{ik} = c_{ik} / w_i, e_i = b_i / w_i$. Since $e^*_{ik} / w_k = \sigma_{ik}$, condition (4) becomes

$$\frac{c_{ik}}{c_{jm}} = \frac{b_i b_k}{b_j b_m}$$

which holds exactly for all variations in prices and income.

(iii) Translog

Only holds locally as in case of AIDS.

Empirical application (Moschini et al):

They tested seven-good demand system with groups: (1) nonfood, (2) fruits & veg., (3) cereals and bakery products, (4) misc. foods, (5) beef and veal, (6) pork, and (7) poultry and fish. Using the RM over sample period 1947-78 they fail to reject weak separability restriction. They perform test using likelihood ratio test $LR = 2[L(\hat{\beta}) - L(\tilde{\beta})]$. (They actually use a "size-corrected" LR test—see p.64 of article.)

<u>Indirect Separability and flexible functional forms (Moschini 2001)</u>

<u>Generalized Composite Commodity Theorem (GCCT)</u>

Hicks-Leontief (HL) composite theorem: define $\rho_i = \log(p_i / P^G)$. Let ρ be the n-vector of ρ_i. The HL theorem states that w, n-vector of group expenditures shares, maximizes utility given p if ρ is constant.

The GCCT allows ρ to vary over time and assumes only that distribution of ρ is independent of p (and of income). That is, changes in relative prices of goods within the group are uncorrelated to the general rate of inflation of the group.

For required behavior of group prices, many commonly used utility functions satisfy conditions for GCCT aggregation including

- (i) All homothetic utility functions.
- (ii) AIDS model.
- (iii) Translog model.
- (iv) Case of two groups.

Arbitrary utility functions with arbitrary number of groups can come close to satisfying the required conditions if certain functions of ρ are small.

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V. Cross-Section Demand, Quality, and Household Production Theory

The traditional approach to including household characteristics in demand functions is as follows (George and King):

$$q_i = g_i(x, p, c)$$

where c = vector of household characteristics. If prices are the same across consumers, then

$$q_i = \bar{g}_i(x, c)$$

or, because prices are assumed constant, this demand function is often represented as

$$p_i q_i = \bar{g}_i(x, c)$$

Given a particular specification for the functional form, equations of this form are then used to estimate Engel curves. Deaton and Muellbauer (chapter 3) point out that it is often the case that the data suggest functions of the following form :

$$w_i = \alpha_i + \beta_i \log x$$

which is nothing more than the AIDS model with prices omitted.
Through comparing time-series estimates with cross-sectional estimates of demand we can infer generally that

(1) time series estimates of demand can be viewed as short-run estimates of income elasticities.

(2) Cross-sectional estimates of demand can be viewed as long-run estimates of income elasticities because the income distribution only changes slowly over time.

Importance of price effects (Cox and Wohlgenant):

(1) Prices are generally not the same across households.

(2) Most products consumed are not elementary goods but are rather composite commodities and therefore prices may vary due to quality effects.

Price variation can occur due to regional differences, price discrimination, through services purchased jointly with goods, seasonal effects, and quality differences from heterogeneous aggregates. If the structure of demand is relatively constant, then seasonal and regional price variation can be attributed to changing supply conditions, which in turn can be used to identify demand curves. Prices can vary from one location to another because of opportunity cost of time and fact that acquiring information about prices at alternative locations is costly.

Quality effects can be evaluated as follows:

Start with the identity,

$$\log x_i = \log p_i + \log q_i$$

$$\frac{\partial \log x_i}{\partial \log x} = \frac{\partial \log p_i}{\partial \log x} + \frac{\partial \log q_i}{\partial \log x}$$

where the first term on the right-hand side is the *quality elasticity* and the second term is the usual income elasticity of demand. For elementary goods, the quality elasticity would be expected to be zero. For commodities (composite goods), the quality elasticity can be different from zero.

Given the importance of prices in cross-sectional demand, the demand model for application with cross-sectional data can be reformulated as follows:

$$q_i = g_i[p_i(x,c), p^-(x,c), x, c]$$

where p^{-} denotes the price vector with the ith price omitted. The effects of household income and characteristics on demand are as follows:

$$\frac{\partial q_i}{\partial x} = \frac{\partial g_i}{\partial p_i}\frac{\partial p_i}{\partial x} + \sum_{j \neq i}\frac{\partial g_i}{\partial p_j}\frac{\partial p_j}{\partial x} + \frac{\partial g_i}{\partial x}$$

$$\frac{\partial q_i}{\partial c_r} = \frac{\partial g_i}{\partial p_i}\frac{\partial p_i}{\partial c_r} + \sum_{j \neq i}\frac{\partial g_i}{\partial p_j}\frac{\partial p_j}{\partial c_r} + \frac{\partial g_i}{\partial c_r}$$

where c_r denotes a specific element of the vector of household characteristics (e.g., household size).

What these derivatives show is that the impact of income (household characteristic) on demand for a given commodity in general cannot be determined through simple differentiation of the Engel function. Prices first need to be adjusted for quality effects induced by income changes (household characteristics) to find the pure income (household characteristic) change on demand.

General approach (Hanneman):

The consumer is viewed as maximizing utility as a function of a vector quantities of the commodities (q), a vector of commodity specific characteristics ($b_i = (b_{i1},...,b_{ik_i})'$, a composite good ($z$) representing other good, and c the vector of household characteristics. Also let α_i = the quantity price of the ith good and $\gamma_i = \gamma_i(\gamma_{i1},...,\gamma_{ik_i})'$ be the vector of quality prices.

The consumer's choice problem is to

Maximize $\quad u = v(q,b,z,c)$

Subject to: $\sum_i p_i(b_i)q_i + z = x$

where $p_i = \alpha_i + \sum_j \gamma_{ij} b_{ij}$. The solutions to this optimization problem (assuming interior solutions) are

$$q_i = g_i^q(\alpha, \gamma, c, x)$$

$$b_{ij} = g_{ij}^b(\alpha, \gamma, c, x)$$

In empirical applications, the focus has been on ways to separate quality components from quantity components of prices. Hedonic regressions of the form $p_i = \alpha_i + \sum_j \gamma_{ij} b_{ij} + \varepsilon_i$ are estimated and prices adjusted for the estimated quality effects, $p_i - \sum_j \hat{\gamma}_{ij} b_{ij}$, are then used as regressors in the demand model. Complications in estimation are introduced by the existence of zero values for consumption.

<u>Household Production Model (Deaton and Muellbauer; Stigler and Becker)</u>

For this model, the individual is viewed as both a consumer and producer. The individual is assumed to derive utility from *m* non-market (primary) commodities $z_1, ..., z_m$. Each z_i is produced according to the production function

$$z_i = f_i(q_{1i}, ..., q_{ni}, l_{0i}; k) \qquad (1)$$

where q_{ji} = quantity of the *jth* produced market good consumed by commodity *i*, l_{0i} = amount of household time consumed by the *ith* primary commodity, and k is the vector of fixed inputs.

The optimization problem of the individual is to

$$
\begin{aligned}
&\text{Maximize} && u = v(z_1, \ldots, z_m) \\
&\text{Subject to:} && z_i = f_i(q_{1i}, \ldots, q_{ni}, l_{0i}; k), \\
& && \sum_j q_{ji} = q_j, \\
& && \sum_i l_{0i} = l_0, \\
& && x = wT + \mu = \sum_j p_j q_j + wl_0,
\end{aligned}
\tag{2}
$$

where T is total number of hours available, w is the wage rate, and μ is non-labor income.

Problem (2) can be solved in two stages. In the lower stage, the individual minimizes costs subject to the production function (1). In the upper stage, the individual maximizes utility subject to the cost function obtained from the lower stage optimization problem. This upper stage optimization problem is often characterized as

$$
\begin{aligned}
&\text{Maximize} && u = v(z) \\
&\text{Subject to:} && x = \sum_j \pi_j^0 z_j \\
& && \pi_j = \pi_j(w, p, z^0; k)
\end{aligned}
$$

where z^0 denotes the optimal solution. The functions, π_j, are marginal cost functions obtained from differentiating the cost function, $c(w, p, z^0; k)$ with respect to z_j. The solutions to this upper stage problem are

$$
z_j = z_j(x, \pi^0)\,. \tag{3}
$$

The solutions to the household production model expressed in the form in (3) indicate that it is shadow prices of the commodities that are the relevant prices that influence consumer demand. These shadow prices depend upon prices of the purchased market goods, wage rate or opportunity cost of time, and household production capital. The problem formulated in terms of upper and lower optimization problems also makes clear that demand for market goods are really derived demand functions. To see this, note that application of Shephard's lemma to the cost function, $c(w, p, z^0; k)$, yields demand functions for market goods:

$$q_j = q_j(w, p, z^0; k) \tag{4}$$

Substitution of (3) into (4) would then produce derived demand functions that depend on market good prices, the wage rate, shadow prices of commodities, income, and household production capital. Note that the shadow prices in equation (3) are endogenous and depend upon the optimal value of the commodity vector, unless the production functions in (1) exhibit constant returns to scale.

<u>Quality Variation, Prices, and Consumer Demand</u>

The purpose of this section is to evaluate the theoretical underpinnings of hedonic regressions. A set of hedonic prices

$$p(z) = p(z_1,..., z_n), \tag{1}$$

where $z_i =$ amount of the *ith* characteristic in each commodity, is determined by certain market clearing conditions: amounts of commodities offered by sellers at every point in the plane must equal amounts demanded by consumers choosing to locate there. Both producers and consumers base their locational and quantity decisions on maximizing behavior, and equilibrium prices are determined such that buyers and sellers are perfectly matched. Market-clearing prices, $p(z)$, are determined by distribution of consumer tastes and producer costs. Rosen (1976) shows how to recover the underlying distribution of parameters.

Market equilibrium:

Assumptions:

1. The components of z are objectively measured.
2. Sufficiently large number of differentiated products available so that z is continuous.
3. Each product has quoted market price, $p(z)$, associated with fixed value of vector z.
4. $p(z)$ is increasing in all of its arguments (i.e., each z_i is treated as a "good".).
5. $p(z)$ possesses continuous second order derivatives.
6. Products are sold in separate, though interrelated, markets.

The individual's consumption decision is to:

Maximize $\quad u = v(x, z)$

Subject to: $\quad y = x + p(z)$

In this specification, x represents all other goods and the assumption is made that the consumer buys one unit of the product. The F.O.N.C. for utility maximization are

$$\frac{\partial p}{\partial z_i} = p_i = \frac{v_{z_i}}{v_x}, \qquad i=1,\ldots,n$$

Define the bid function $\theta(z;u,y)$ according to

$$v(y-\theta,z)=u \tag{2}$$

Upon differentiation of (2) we can establish that θ is increasing in z_i at a decreasing rate. The function $\theta_{z_i} = p_i = \dfrac{v_{z_i}}{v_x}$ >0 is the marginal rate of substitution between z_i and money, or the implicit marginal valuation the consumer places on z_i at a given utility level and income. It is the reservation demand price for an additional unit of z_i, which is decreasing in z_i.

Production decision:

Maximize $\quad \pi = mp(z)-c(m,z)$

where *m* is the number of units produced. The F.O.N.C. for profit maximization are

$$p_i(z)=c_{z_i}(m,z)/m, \qquad i=1,\dots,n \tag{3}$$

$$p(z)=c_m(m,z) \tag{4}$$

At the optimum design, marginal revenue from additional attribute equals marginal cost of production per unit sold. Quantities are produced up to the point where unit revenue

The offer function, $\phi(z;\pi)$, shows the unit prices (per model) firm is willing to accept at constant profit. This function is obtained by solving (4) for $m=m(\phi,z), \phi=p(z),$ and using this result in the profit equation, $\pi = mp(z)-c(m,z)$, to eliminate *m*.

We can establish like before with consumer choices the slope of ϕ and curvature condition. For given ϕ, $c_{z_i z_i}>0$ so that ϕ is increasing at an increasing rate in z_i.

What do hedonic prices mean? Generally speaking, observations $p(z)$ represent a joint envelope of a family of value functions and another family of offer functions. We can think of the θ_{z_i} 's as reservation demand prices at constant utility. These functions therefore represent inverse compensated demand functions. The function ϕ_{z_i} is the reservation supply price of an incremental unit of z_i and reflects a profit-compensated inverse supply function for characteristic z_i. Thus, as shown in the attached figure, the observed marginal prices merely connect equilibrium prices and characteristics and reveal little about the underlying supply and demand functions.

95a

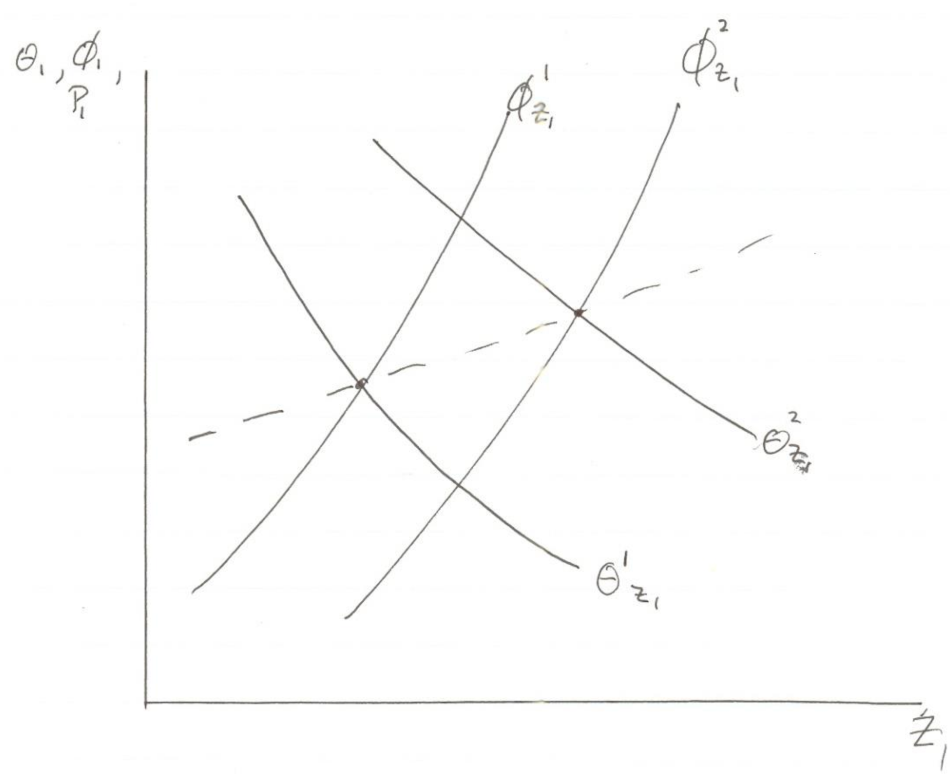

Illustration of Market Equilibrium and Identification Problem

How can we identify the parameters of the demand and supply functions? Structurally, the market structure can be represented as follows:

$$p_i(z) = F^i(z, y_1) \qquad \text{(demand)}$$

$$p_i(z) = G^i(z, y_2) \qquad \text{(supply)}$$

First, estimate $p(z)$ by the usual hedonic method without regard to the supply and demand shift variables. Second, compute the set of implicit marginal prices, $\partial p(z)/\partial z_i = \hat{p}_i(z)$ for each buyer and seller. Finally, use each $\hat{p}_i(z)$ as endogenous variables in the above set of supply/demand equations and apply standard estimation methods for simultaneous equations. There are two special cases in this instance:

(a) If cost conditions identical across firms, then y_2 drops out of second equation and $\hat{p}(z)$ is the offer function.

(b) If buyers are identical, but sellers differ, then y_1 drops out of the first equation and cross-sectional observations identify the compensated demand functions.

Quality Variation and Consumer Demand (Nelson 1991)

Consider the Houthakker-Theil problem:

Maximize $\quad u = v(q_1, q_2, \ldots q_G, \ldots, q_m, v_1, v_2, \ldots, v_G, \ldots, v_m)$

Subject to: $\quad \sum_{G=1} p_G q_G = y$

where $q_G = \sum_{i \in G} x_i$.

Problems with this approach:

(a) Inherent ambiguity in how q_G relates to quantity demanded of consumer demand. Quantity demanded is a function not only of prices and income but also quality choice.

(b) Use of physical quantities involves selection of one unit of physical measurement from long list of possibilities.

(c) Without further assumptions, how do q_G's relate to any item of real interest.

(d) Model is difficult to solve in general form.

Quality and theoretically valid aggregation:

Maximize $u = v(x_1, x_2, ..., x_R)$

Subject to: $\sum_{i \in R} p_i x_i = y$

where x_i is quantity of the *ith* elementary good. Controlling for quality variation is *equivalent* to the problem of grouping the elementary goods. Let $Q_G (G = 1,..., M)$ and $P_G (G = 1,..., M)$ be the composite commodity aggregates and prices so that solving

Maximize $u = v(Q_1, ..., Q_G, ..., Q_M)$

Subject to: $\sum_{G=1}^{M} P_G Q_G = y$

is equivalent to solving the disaggregate problem.

What are the requirements for aggregation? Recall that homogeneous separability or strong separability are required to treat allocation among the Q_G's as functions of P_G's and y. Each composite commodity must be a linear homogeneous function of its elementary goods if $P_G Q_G = \sum_{i \in G} p_i x_i$. When will this occur?

Case I: $Q_G = \sum_{i \in G} x_i$.

The elementary commodities are perfectly substitutable. Therefore, any discussion about quality differences is irrelevant. This specification only works if the goods are homogeneous or very nearly homogeneous.

Case II: Preferences are weakly separable and within-group preferences are homothetic. This means, for example, that at constant prices, the ratio of hamburger to steak must equal the same for rich as well as poor consumers, at constant prices. Even if true, this specification is less restrictive than case I and fails to justify use of simple sum of quantity demanded as a measure of demand.

Case III: With restrictions on relative prices, within group aggregation can provide justification for use of both composite commodities and clear definition of quality. Assume $p_i = P_G p_i^*$ where p_i^* is the base price of good *i* and P_G is the factor of proportionality. By Hicks composite commodity theorem

$$Q_G = \sum_{i \in G} p_i^* x_i \tag{1}$$

$$E_G = \sum_{i \in G} p_i x_i = \sum_{i \in G} (P_G p_i^*) x_i = P_G \sum_{i \in G} p_i^* x_i = P_G Q_G \tag{2}$$

"Unit values" are calculated as

$$V_G = E_G / q_G = P_G Q_G / q_G \tag{3}$$

where $q_G = \sum_{i \in G} x_i$. Theil and Cramer define average quality within a group as

$$v_G = \sum_{i \in G} (x_i / q_G) p_i^* = \sum_{i \in G} p_i^* x_i / q_G \tag{4}$$

The following identities hold:

$$Q_G = v_G q_G \tag{5}$$

$$E_G = P_G v_G q_G \tag{6}$$

$$V_G = P_G v_G \tag{7}$$

Income and price elasticities for the composite goods are computed as follows:

$$e_{QI} = e_{vI} + e_{qI} \tag{8}$$

$$e_{EP} = e_{vp} + e_{qp} \tag{9}$$

Because $\partial \log P / \partial \log y = 0$ and $\partial \log P / \partial \log P = 1$,

$$e_{EI} = e_{QI} \tag{10}$$

$$e_{EP} = 1 + e_P \tag{11}$$

$$e_{VI} = e_{vI} \tag{12}$$

$$e_{VP} = 1 + \theta_{vp} \tag{13}$$

Equations (8) and (9) imply the income elasticity and own-price elasticity of composite commodity are sums of quality and physical sum elasticities. Equations (10) and (11) indicate that income and price elasticities could also be derived from expenditure elasticities when composite commodity quantity is measured correctly.

Summary:

(a) Simple sum aggregates appropriate only for homogeneous goods.

(b) Hick's composite commodity theorem permits aggregation of elementary goods consistent with freely variable choices of consumer across elementary goods with varying characteristics.

(c) Hick's assumption yields analogous equations for implicit quantity, $Q_G = E_G / P_G$, as does homogeneous separability.

(d) $q_G = E_G / V_G$ only legitimate measure of quality if unit values are actually exogenous prices (i.e., if quality effects are absent).

<u>Equivalence scales and increasing price variation (Lewbel 1989)</u>

<u>Modeling demand with product differentiation: specification and estimation issues</u>

$$q_i = g_i(x, p_1, \dots, p_n)$$

$n^2 - \frac{n(n-1)}{2} - n - 1$ independent restrictions, which is too many to estimate when *n* in large.

Alternative approaches in this context:

(1) Use separability and multi-stage budgeting.
(2) Use GCCT to aggregate goods.
(3) Discrete choice modeling
    (a) multinomial logit
    (b) random coefficient model
    (c) nested logit model
(4) Discrete metric modeling approach

Discrete modeling approach:

Advantages:
(1) Avoids having to estimate all elasticities
(2) Resulting demands can make predictions about demand for new products
(3) Allows one to move easily from statements about aggregate demand to statements about consumer utility.
Disadvantages:
(1) Rules out purchases of multiple items.

(2) Does not easily accommodate dynamic aspects of demand.

(3) Typically places too strong of restrictions on substitutability.

General Approach (Berry):

Let i equal consumer and j equal product. The utility function is assumed to have the form:

$$U_{ij} = Y_{ij}\beta + \epsilon_{ij}$$

where Y_{ij} is the set of relevant characteristics (including price) for consumer i – product j combination.

Consumer i chooses one unit of product j if

$$U_{ij} > U_{ik} \ \ \forall j \neq k$$

If ϵ_{ij} has extreme value distribution, McFadden shows that

$$\pi_{ij} = \frac{\exp(Y_{ij}\beta)}{\sum_{k=1}^{K} \exp(Y_{ik}\beta)}$$

This expression if often interpreted as market share of firm i product j. The vector β represents individual tastes that are not observable but estimable. The error term ϵ_{ij} represents unobservable taste parameters that are not estimable.

The multinomial model suffers from a major problem: assumption of independence of irrelevant alternatives (IIA) which implies that cross-price elasticities of demand are equal.

Ex. Let $Y_{ij}\beta = \alpha P_j + Z_{ij}\gamma$. Omitting i for the moment, we can show that

$\frac{\partial \pi_j}{\partial P_k}\frac{P_k}{\pi_j} = -\alpha P_k \pi_k = \frac{\partial \pi_k}{\partial P_j}\frac{P_j}{\pi_k}$.

In light of this problem, Berry introduces the random parameter model:

$$U_{ij} = X_j\overline{\beta_\iota} - \alpha P_j + \xi_j + \epsilon_{ij}$$

where $\overline{\beta_{\iota k}} = \beta_k + \sigma_k \varsigma_{ik}, \quad \varsigma_{ik}\ i.i.d.$ with mean zero,

$$X_j\overline{\beta_\iota} = \sum_k \overline{\beta_{\iota k}} x_{jk} = \sum_k \beta_k\, x_{jk} + \sum_k \sigma_k \varsigma_{ik}\, x_{jk}$$

This expression implies that the utility function is

$$U_{ij} = X_j\beta - \alpha P_j + \xi_j + \epsilon_{ij} + \sum_k \sigma_k \varsigma_{ik}\, x_{jk} = X_j\beta - \alpha P_j + \xi_j + v_{ij}$$

The mean level of utility is: $\delta_j = X_j\beta - \alpha P_j + \xi_j$

so that

$$U_{ij} = \delta_j + v_{ij}$$

By definition,

$$E(s_{ij}) = \int_{-\infty}^{\infty} \frac{\exp(Y_{ij}\beta)}{\sum_{k=1}^{K} \exp(Y_{ik}\beta)}\, dF(\beta)$$

Empirically, this expression usually needs to be evaluated numerically. The approach to estimation is simulated method of moments (SMOM).

Outline of method:

(i) The consumer chooses good j if $U_{ij} > U_{ik} \ \ \forall j \neq k$.

(ii) The choice set is $A = \{\delta_j + v_{ij} \geq \delta_k + v_{ik}\ , \forall j \neq k\}$

(iii) $s_j(\delta_j) = \int_A \ dF(v)$

where $F(v)$ is the cumulative distribution function – see, e.g., Train (2000).

Application (Tchetehik, Fleischer, and Finkelstein 2009):

Assume $v_{ij} \sim i.i.d.$ as extreme value function. Then

$$s_j(\delta_j) = \frac{\exp(\delta_j)}{1 + \sum_{k=1}^{K} \exp(\delta_j)}$$

Two-stage process:

1. Choose region.
2. Choose accommodation within region.

Assume there exists and outside good,

$$s_0(\delta_j) = \frac{1}{1 + \sum_{k=1}^{K} \exp(\delta_j)}\,, \quad \delta_0 = 1$$

Therefore,

$$\log\left(\frac{s_j}{s_0}\right) = X_j\beta - \alpha P_j + \xi_j$$

Because P_j and ξ_j are correlated, we have simultaneity bias so GMM is used in estimation.

Multi-stage budgeting approach:

Ex. Hausman, Leonard, and Zona estimate demand for beer at brand level taking this approach.

Brand level:

$$S_{int} = \alpha_{in} + \beta_i \log\left(\frac{Y_{nGt}}{P_{nt}}\right) + \sum_{1}^{J} \gamma_{ij} \log P_{jnt} + \varepsilon_{int}$$

where Y_{nGt} is overall segment expenditure, S_{int} is market share of brand i in segment, *n* denotes a given city, and *t* the time perod.

The market segments are premium, light, and popular.

Middle level:

$$\log q_{mnt} = \beta_m \log Y_{nGt} + \sum_{1}^{M} \delta_k \log \Pi_{knt} + \varepsilon_{mnt}$$

Upper level:

$$\log u_t = \beta_0 + \beta_1 \log Y_t + \log \Pi_t + Z_t \delta + \varepsilon_t$$

In estimating brand level demand, there's concern about correlation between brand level prices and error terms (why?)

Answer: Imperfect competition implies correlation of unobserved demand components with prices.

Insturmental variables used: For a given product in a given city they use prices in other cities as instruments for prices.

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VI. Derived Demand and Price Spreads (Marketing Margins)

So far we have focused primarily on consumer demand. Of equal, or perhaps more, importance is derived demand. This is especially the case for food commodities because we typically only have measures of quantities of food at the wholesale or farm level. Also, for agricultural commodities (a) agricultural policies have focused on returns and prices to farmers, and (b) public policy concerns have historically been about size and change in price spread between retail and farm prices. Retail and farm (agricultural raw material) prices are linked by unit marketing costs (labor, packaging, transportation, energy, etc.). This implies that an understanding of the nature and behavior of price spreads (marketing margins) is crucial to our understanding of demand behavior at the farm level.

The most common definition of the marketing margin is that the farm-retail price spread is the difference between the retail price of a given product and its farm value, the payment to farmers of an equivalent quantity of the farm product (excluding any by-product values). In essence, the price spread definition is the retail price less the unit farm value.

Traditional Approach to Modeling Marketing Margin:

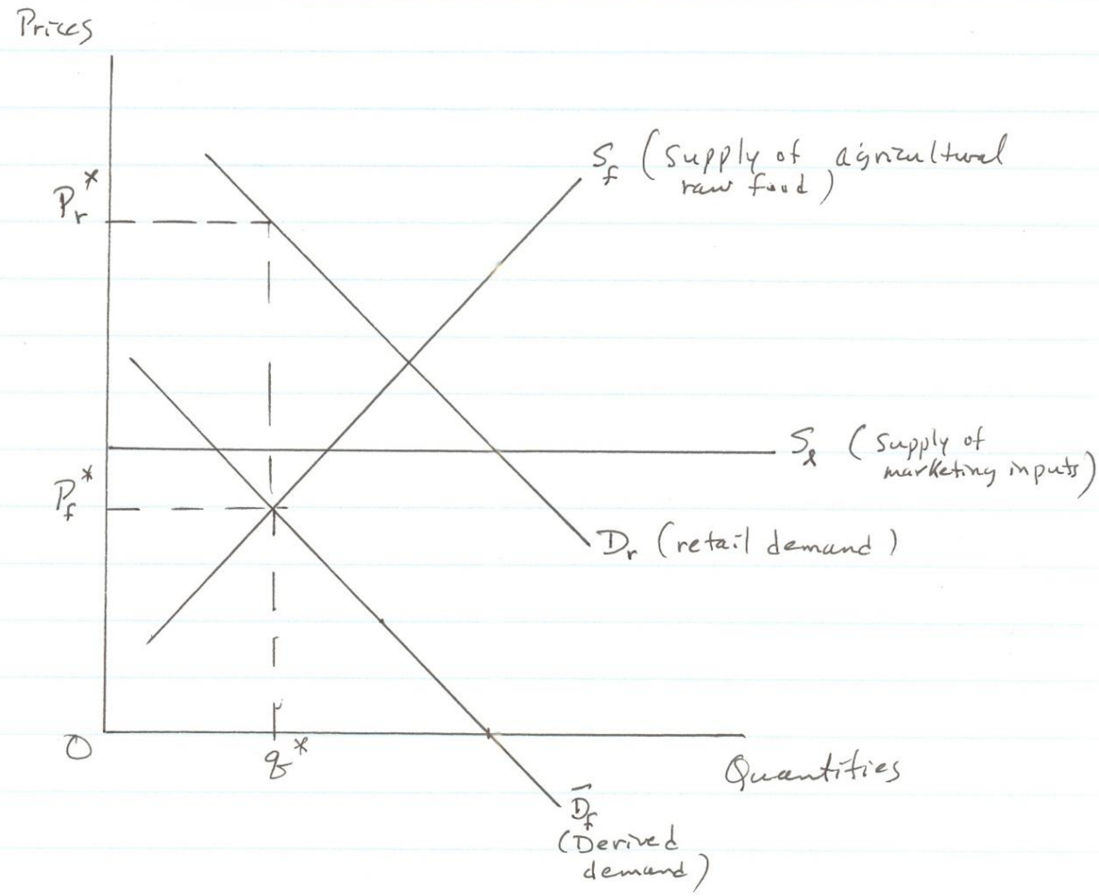
Consider the theory of joint demand where demand for the final product (retail commodity) is viewed as a joint demand for all the factors of production. When the final product is produced with fixed proportions of all the inputs, a derived demand schedule for a given factor can be obtained by subtracting per-unit costs of all other factors from the demand price for the final product (see attached figure). In the figure, p_r, p_f, q represent retail price, farm price (unit farm value), and quantity of product (assumed to be the same at both farm and retail level). Given demand for the final product, derived demand for the farm product, and supply of the farm product, equilibrium retail and farm prices are determined as shown. The difference between the equilibrium retail price and farm price represents the marketing margin for the given food product.

There are three possible ways the price spread or marketing margin can change as quantity of the farm product (from shifts in the farm product supply curve) changes. The three possible types of margin behavior depend on the shape of the supply curve of the supply curve of the composite marketing input:

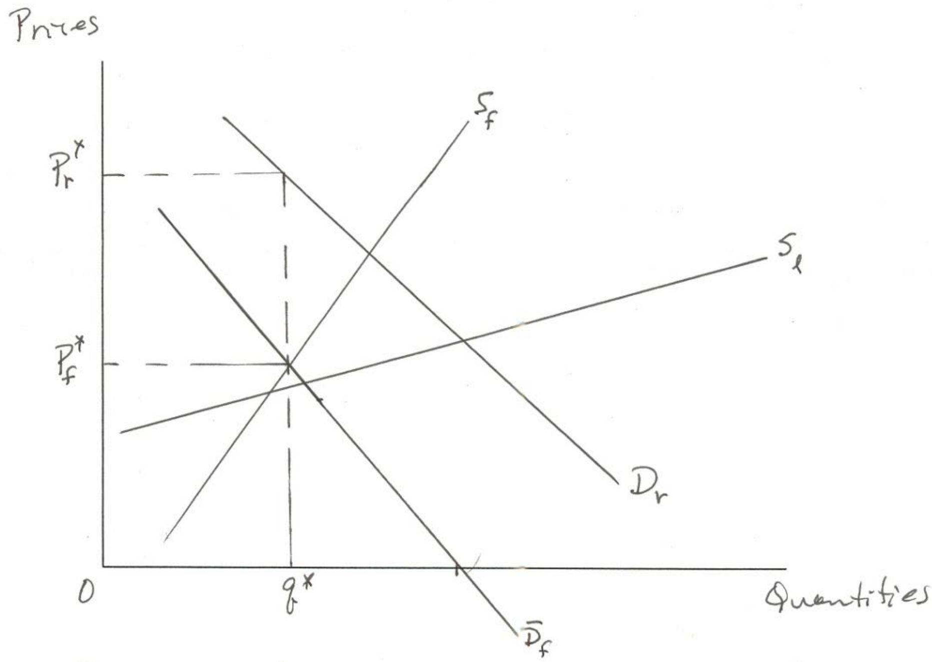
(a) Horizontal supply curve S_l implies $m = p_r - p_f$ does not change as either q or p_r changes (holding D_r and S_l curves stationary).

(b) Upward sloping supply curve S_l implies m is positively related to q or negatively related to p_r.

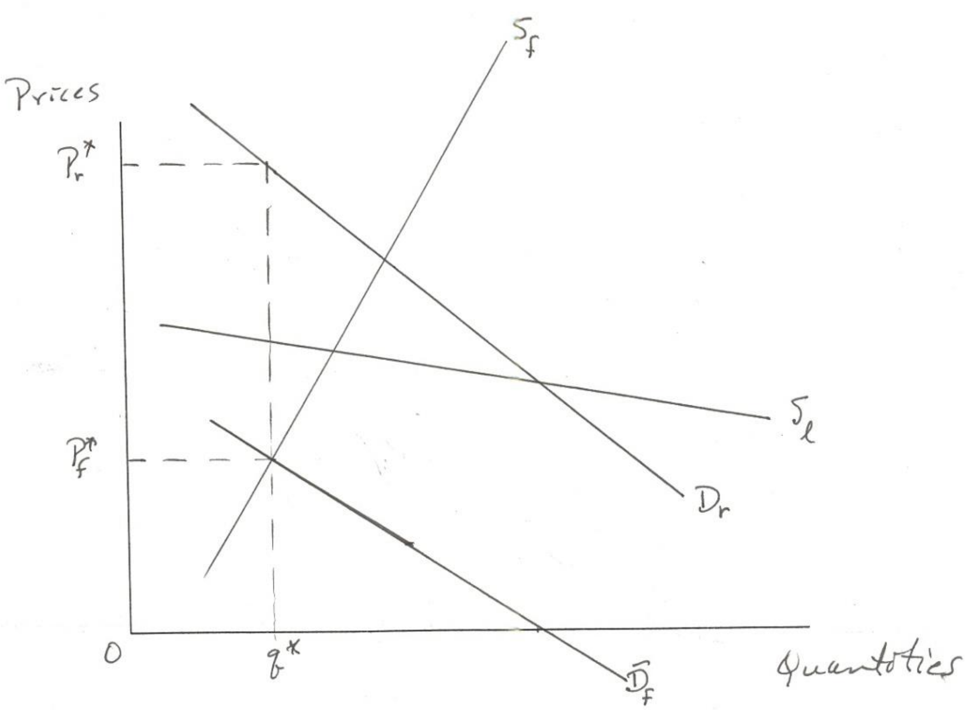
(c) Downward sloping supply curve S_l implies m is negatively related to q or positively related to p_r.



Market Equilibrium of Retail and Farm Prices with Fixed Factor Proportions



$\underline{S_\ell}$ upward sloping



$\underline{S_\ell}$ downward sloping

The empirical evidence is most consistent with case (c). This margin behavior is not consistent with short-run competitive pricing behavior and would only occur in the long-run in a decreasing cost industry. Given the prevalence of this behavior among food commodities over different time periods, it seems reasonable to search for other explanations of this type of behavior. Two other possibilities are:

(a) Imperfect competition, and

(b) Substitution between the agricultural food input and the composite marketing input.

We will examine each one of these possibilities later, but for now will simply look at what the three types of margin behavior we have identified imply about the relationship between retail demand elasticities and derived demand elasticities. Assume for sake of simplicity that the margin behavior can be specified as a linear function of retail price:

$$m^j = \alpha^j + \beta^j p_r^j \tag{1}$$

This specification is based on the fact that many studies by the U.S.D.A. and other researchers over many years and many commodities have demonstrated that margins are neither constant absolute nor constant percentages (of retail prices), but are somewhere in between the two. Constant percentage and constant absolute price spreads are special cases of (1):

Constant percentage: $m^j = \beta^j p_r^j$ (2a)

Constant absolute: $m^j = \alpha^j$ (2b)

Given this specification of price spread behavior, formulas for derived demand elasticities can be obtained. By definition, the derived demand elasticity for the *ith* farm product with respect to the price of the *jth* farm product is

$$E_{ij} = \frac{\partial q_f^j}{\partial p_f^j} \frac{p_f^j}{q_f^i} \tag{3}$$

If the production function is Leontief (which is the assumption underlying this margin behavior) so $q_f = q_r = q$, then

$$E_{ij} = \frac{\partial q^i}{\partial p_f^j} \frac{p_f^j}{q^i} \tag{4}$$

From the definition of the price spread in (1), recalling $m^j = p_r^j - p_f^j$, we have

$$p_r^j = \frac{\alpha^j}{1-\beta^j} + \frac{1}{1-\beta^j} p_f^j \tag{5}$$

Applying the chain-rule theorem yields

$$\frac{\partial q^i}{\partial p_f^j} = \frac{\partial q_r^i}{\partial p_r^j} \frac{\partial p_r^j}{\partial p_f^j} = \frac{\partial q^i}{\partial p_r^j} \frac{\partial p_r^j}{\partial p_f^j}$$

Using this result and the definition from (4) we can express the derived demand elasticity as follows:

$$E_{ij} = \frac{\partial q^i}{\partial p_f^j} \frac{p_f^j}{q^i} = \frac{\partial q^i}{\partial p_r^j} \frac{p_r^j}{q^i} \frac{\partial p_r^j}{\partial p_f^j} \frac{p_f^j}{p_r^j} = e_{ij}\eta_j \tag{6}$$

where recall that e_{ij} is the Marshallian demand elasticity and where η_j is defined as the elasticity of price transmission from the farm to the retail price. Therefore, in the case of fixed factor proportions, we have shown that the derived demand elasticities can be obtained as products of retail demand elasticities and elasticities of price transmission.

What does (6) imply about the relationship between the elasticity of demand at the retail level and elasticity of demand at the farm level? Given the relationship defined by (1) the answer is $|e_{ij}| \geq |E_{ij}|$. This can be established by noting that when margin behavior is that indicated by (2a), $E_{ij} = e_{ij}$, and when margin behavior is indicated by (2b), $|E_{ij}| = |e_{ij} p_f^j / p_r^j| < |e_{ij}|$, provided that the farm price is less than the retail price.

Gardner (1975) shows that the relationship above is conditional on the assumption of fixed proportions. If input substitution can occur it is possible $|e_{ij}| < |E_{ij}|$.

Effect of imperfect competition:
The general condition for profit maximization of a firm is marginal revenue equals marginal cost which can be written as follows:

$$MR = p_r\left(1 + \frac{1}{e}\right) = MC$$

where e is the elasticity of demand facing the firm, and where $MC = p_f + MC_p$, where MC_p is marginal cost of processing, assumed here to be constant for sake of simplicity. Using the above relationship when $m = p_r - p_f$ we find that the price spread is related to retail price as follows:

$$m = MC_p - \frac{1}{e} p_r$$

Therefore, the price spread and retail price will be positively related when e is constant, i.e., when demand facing the firm is a constant elasticity demand function.

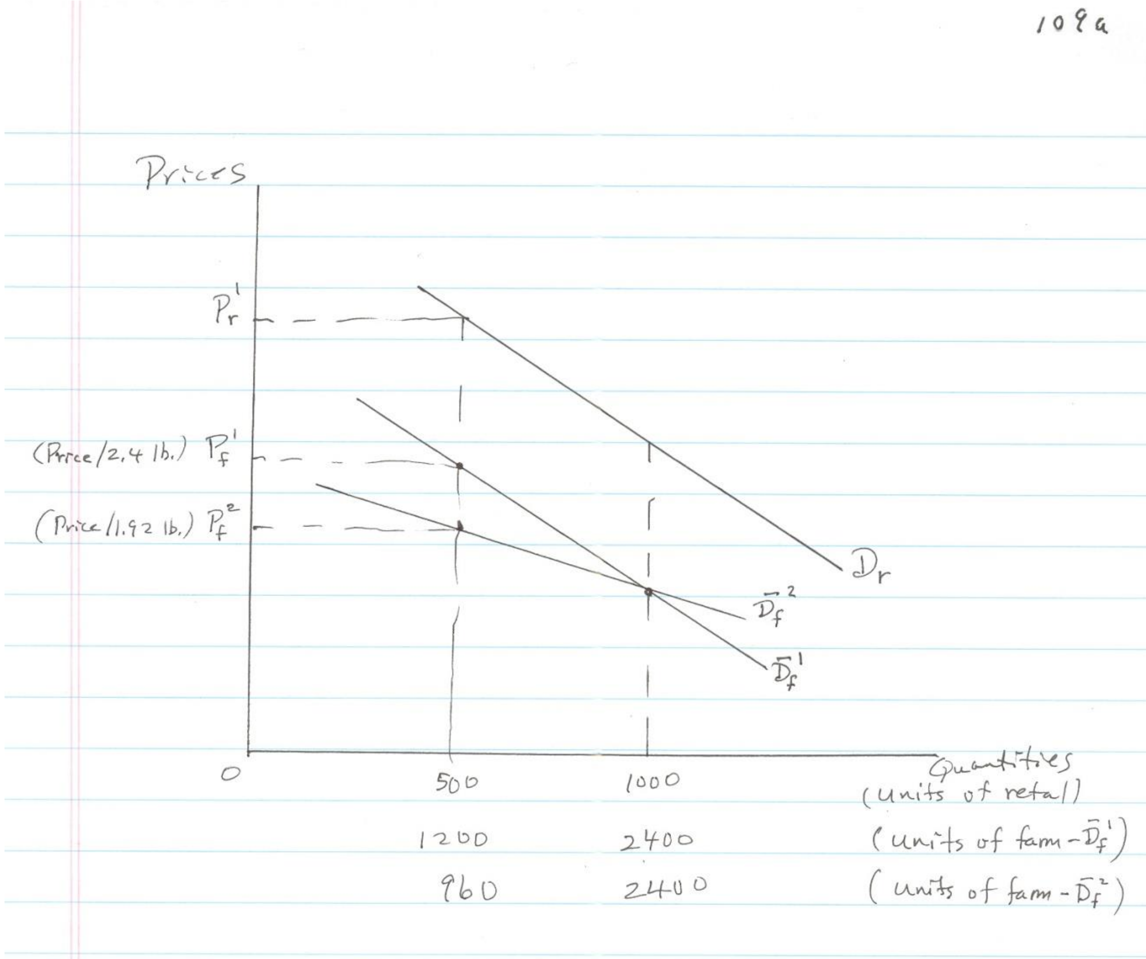
What if the demand function is not of the constant elasticity variety, for example it is linear? In this case, profit maximization implies $p_r = 0.5MC + const$, so that

$$m = const - p_r$$

Therefore, for a linear demand curve, the price spread and retail price would be negatively related. This means that the effect of imperfect competition in the output market on the relationship between the price spread and retail price is ambiguous.

Effect of input substitutability:

To show that the existence of input substitutability can result in a positive relationship between the price spread and retail price, consider a specific example: the market for U.S. beef. USDA computes conversion factors from farm to retail weight for various food commodities. In the case of beef, the conversion factor used is 2.4 pounds of live weight per pound of retail weight. This conversion factor can change over time as the nature of the product produced changes and as the amount of fat trimmed by the butcher changes. Indeed, as discussed below, other factors can also cause this ratio to change. Let's assume that associated with a given increase in the farm price (resulting from a decrease in the supply of cattle) that indeed input substitution does occur so that now the conversion factor drops from 2.4 to 1.92. The effect of this input substitution is shown in the attached diagram. All units on the horizontal scale are retail equivalent units. So if there is no input substitution permitted, 1000 units of the retail product would require 2400 units of the raw material and 500 units of the retail product would require 1200 units of the raw material. With input substitution permitted, while 2400 units of the raw material would be required to produce 1000 units of the retail product at the initial price, now only 960 units of the farm product would be required to produce 500 units of the retail product. Note that the effect of input substitution is the cause the demand price of cattle for the 500 pounds of retail product equivalent to now decrease from p_f^1 to p_f^2. This is because, even though the price per unit of farm product does not change, the price per unit of retail product must fall because now the retail product requires less of the farm product at this point due to substitution of other inputs for the farm product. Therefore, we observe a derived demand schedule that is flatter than the retail demand curve, implying a positive relationship between the price spread and retail price for a stationary demand curve of the retail product.



Effect of input substitutability on price spread.

What are the possible sources of substitutability in this case? It is important to understand (as we have already discussed) that the best measure of quantity, and consequently the input-output ratio, is a value measure where individual goods within the composite commodity (which virtually goods are at some level) are (constant) dollar weighted sums of the individual components of the composite good. Therefore, a complete list of sources of input substitutability would include the following:

(1) Reduction in wastage and spoilage (from the farm price increase).

(2) Availability of alternative technologies to produce the retail product.

(3) Differences in efficiencies among firms.

(4) Inter-product or inter-industry input substitution.

(5) Substitution of quality for quantity.

(6) Substitution of household time for purchased raw materials.

Demand for Farm Output as a Factor of Production

In light of the problems we have discussed with the traditional framework in which marketing group behavior is assumed to captured entirely by marketing margins, it is instructive to view derived demand for the farm product as a factor of production in the producing the retail product. Conceptually, retail price is best viewed as the outcome of the interaction of retail demand and retail supply; likewise, farm price is best viewed as the outcome of the interaction of farm level demand and supply.

In this section, we first review the concept of derived demand at the farm level and then discuss derived demand at the industry level. The review follows closely the development of Wohlgenant and Haidacher (1989, Appendix A).

Firm level derived demand:

There are three types of demand functions: (a) output constant, (b) output price constant, and (c) output price variable. Output constant demand functions are the outcome of the following cost minimization problem of the firm:

$$c(q_r, p_f, w) = \min[p_f q_f + wl : q_r = f(q_f, l)]$$

the solutions of which are

$$\begin{aligned} q_f &= \frac{\partial c(q_r, p_f, w)}{\partial p_f} = D_f^c(q_r, p_f, w) \\ l &= \frac{\partial c(q_r, p_f, w)}{\partial w} = D_l^c(q_r, p_f, w) \end{aligned} \tag{1}$$

As in the case of the consumer expenditure function, certain restrictions on these demand functions flow from the properties of the firm's cost function. These include

(1) Homogeneity: input demand functions homogeneous of degree zero in input prices.

(2) Symmetry: cross-partial derivatives of the demand functions with respect to prices are equal.

(3) Negativity: input demand curves are downward sloping.

The second type of input demand functions, output price constant demand functions, are obtained as solutions from the profit maximization of the firm:

$$\pi(p_r, p_f, w) = \max[p_r q_r - p_f q_f - wl : q_r = f(q_f, l)]$$

with

$$\begin{aligned} q_r &= \frac{\partial \pi(p_r, p_f, w)}{\partial p_r} = S_r(p_r, p_f, w) \\ q_f &= -\frac{\partial \pi(p_r, p_f, w)}{\partial p_f} = D_f(p_r, p_f, w) \\ l &= -\frac{\partial \pi(p_r, p_f, w)}{\partial w} = D_l(p_r, p_f, w) \end{aligned} \tag{2}$$

Again, as in the case of the output constant factor demand functions, certain properties of the demand functions flow from the properties of the profit function. In particular,

(1) Homogeneity: output supply and input demand functions are homogenous of degree zero in output price and all input prices.

(2) Symmetry: supply and demand functions are all symmetric with respect to input and output price changes.

(3) Supply curves are upward sloping and input demand functions are downward sloping.

The first two types of derived demand functions can be related to one another by substituting the output supply function from (2) into the output-constant demand functions in (1):

$$\begin{aligned}\hat{D}_f(p_r, p_f, w) &= D_f^c[S_r(p_r, p_f, w), p_f, w] \\ \hat{D}_l(p_r, p_f, w) &= D_l^c[S_r(p_r, p_f, w), p_f, w]\end{aligned} \tag{3}$$

Industry level derived demand:

Although strictly part of the last section, the third type of firm level derived demand function (output price variable demand) are best discussed in this section in the context of industry level derived demand. When all firms in the industry respond to a price change, output price will change. Let $D(p_r)$ denote the output demand function and $\sum S_r^i(p_r, p_f, w)$ denote the aggregate output response of all firms, so that market clearing of the output market is defined as

$$D(p_r) - \sum S_r^i(p_r, p_f, w) = 0$$

Upon implicit differentiation of the market clearing condition, we find that

$$\frac{\partial p_r}{\partial p_f} = \sum \frac{\partial S_r^i}{\partial p_f} \Bigg/ \left[\frac{\partial D}{\partial p_r} - \sum \frac{\partial S_r^i}{\partial p_r}\right] \tag{4}$$

Since profit-maximizing behavior implies that supply curves of individual firms are upward sloping, and assuming that the output demand curve is downward sloping, the denominator of (4) will be negative. In the usual case of normal factors, the numerator will be negative as well so we would expect output price and the factor price to change in the same direction. Designate the solution to (4) as $p_r = p_r(p_f, w)$. Substituting this solution into the factor demand equations in (3) gives the firm's demand functions which take into account the effect of output price always adjusting to equate demand with industry supply:

$$
\begin{aligned}
\overline{D}_f(p_f, w) &= \hat{D}_f[p_r(p_f, w), w] \\
\overline{D}_l(p_f, w) &= \hat{D}_l[p_r(p_f, w), w]
\end{aligned} \tag{5}
$$

Note from (3) that

$$
\frac{\partial \hat{D}_f}{\partial p_f} = \frac{\partial D_f^c}{\partial S_r}\frac{\partial S_r}{\partial p_f} + \frac{\partial D_f^c}{\partial p_f} \tag{6}
$$

and from (5) that

$$
\frac{\partial \overline{D}_f}{\partial p_f} = \frac{\partial \hat{D}_f}{\partial p_r}\frac{\partial p_r}{\partial p_f} + \frac{\partial \hat{D}_f}{\partial p_f} \quad . \tag{7}
$$

Upon substituting (6) into (7) and indexing this response by the *ith* firm we obtain the demand response of the third type of firm level derived demand (output price variable demand):

$$
\frac{\partial \overline{D}_f^i}{\partial p_f} = \frac{\partial D_f^{ci}}{\partial p_f} + \frac{\partial D_f^{ci}}{\partial S_r}\frac{\partial S_r^i}{\partial p_f} + \frac{\partial \hat{D}_f^i}{\partial p_r}\frac{\partial p_r}{\partial p_f} \tag{8}
$$

Therefore, the short-run response of the *ith* firm to a change in factor price consists of three effects: (1) output constant effect, (2) output response effect, and (3) output-price response effect. The first two effects together represent the profit-maximizing response holding output constant and is unambiguously negative. However, the third effect may be either positive or negative, and it is normally expected to be positive. Thus, for an individual firm, the law of demand (with output price variable) can be violated.

However, as proved by Heiner (1982), the ambiguity of the sign of the short-run response disappears when the responses are aggregated over all firms. That is the sign of $\sum \partial \overline{D}_f^i / \partial p_f$ is unambiguously negative. While the proof of this result is tedious (and is not given here—see Heiner for details), the intuition is straightforward. In particular, the industry factor demand curve lies between two demand curves, which are each negatively sloped: (a) the first constructed as the horizontal summation of the constant output demand responses, and (b) the second constructed as the horizontal summation of the output-price constant demand responses. Friedman provides an intuitive discussion of this result. Braulke (1984) shows that this result can be generalized to the case of less than infinitely elastic supply curves for some of the input markets. The significant point of this result is that generally, without any restrictions on input substitutability or diversity among firms, the law of demand holds for industry factor demand when demand for the output is negatively sloped and factor supply curves are upward sloping.

Despite this general result, economists still remain interested in the traditional model of long-run industry behavior where firms are identical and there is free entry and exit into/from the industry. Under these assumptions, there exists an aggregate production function which exhibits constant returns to scale. Constant returns to scale must exist because, with given technology, fixed factor prices, and determinate firm size, a doubling of all industry inputs would be achieved most efficiently by doubling the number of producing firms, which in turn would lead to doubling of industry output. Under these conditions, the industry cost function is linear in output, output price equals minimum average cost of the representative firm, and the relative input demand are independent of the level of output (Silberberg, Diewert 1981). Mathematically, this allows us to write market equilibrium (given exogenous factor prices) as follows:

$$
\begin{aligned}
q_r &= D(p_r), \\
p_r &= uc(p_f, w), \\
q_f &= uc_f(p_f, w)q_r, \\
l &= uc_w(p_f, w)q_r
\end{aligned}
\tag{9}
$$

where $uc(p_f, w)$ represents the (minimum) unit cost of producing the retail output such that $uc(p_f, w)q_r$ represents total industry cost. The first equation in (9) is consumer demand for the retail output, the second equation is the (inverse) output supply function, the third equation is the output-constant demand for the farm input, and the fourth equation is the output-constant demand for the marketing input. Subscripts associated with the unit cost function denote partial derivatives with respect the farm price and wage rate, respectively. Overall, the equations in (9) describe market equilibrium of the processing-marketing sector given exogenous factor prices. If we were to allow for upward sloping supply curves for either the farm input and/or marketing input, we could change the industry from a constant-cost industry into an increasing or decreasing cost industry, depending upon whether expansion of industry output increases or decreases average cost of production. An increasing or decreasing cost industry could also result from non-pecuniary diseconomies or economies, but such a representation of the industry equilibrium would require a modification of the industry production function and, therefore, the industry total cost function.

The equations in (9) can be used to develop quantitative expressions for industry derived demand. Substituting the first two equations into the third equation and differentiating with respect to the farm price gives

$$\frac{\partial q_f}{\partial p_f} = \frac{\partial uc_f}{\partial p_f} q_r + uc_f \frac{\partial D}{\partial p_r} \frac{\partial uc}{\partial p_f}$$

$$
\begin{aligned}
&E_{ff} = E_{ff}^{c} + s_f e \\
&or \\
&E_{ff} = -(1 - s_f)\sigma + s_f e
\end{aligned}
\tag{10}
$$

where E_{ff}^{c} is the elasticity of demand for the farm input with respect to farm price, holding industry output constant, s_f is the farm value share, and σ is the elasticity of substitution between the farm input and marketing input.

<u>Comparative Statics of Farm and Retail Prices in a Competitive Food Industry (Identical Firm Case):</u>

$$
\begin{aligned}
&q_r = D(p_r, z), \\
&p_r = uc(p_f, w), \\
&q_f = uc_f(p_f, w)q_r, \\
&l = uc_w(p_f, w)q_r, \\
&p_f = g(q_f, c), \\
&w = h(l, t)
\end{aligned}
\tag{1}
$$

These six equations describe market equilibrium in the retail market for food, the market for the farm output, and the market for marketing inputs consumed by the food processing-marketing industry. The first four equations represent demand and supply for food and demand for the farm output and marketing inputs; the last two equations represent (inverse) supply functions for the farm output and marketing inputs. The variables z, c, and t represent retail demand shifters, supply shifter of the farm input, and supply shifter of the marketing input, respectively.

These six equations will produce equilibrium values of retail price and quantity, farm price and quantity, and marketing input and quantity for given values of the retail demand and input supply shift variables. The specification could also be modified to include exogenous shifters of the cost function but this modification is ignored here.

In order to derive the comparative statics of changes in demand and supply on retail and farm prices it is useful to first derive the comparative statics for farm price and the price of marketing inputs and then find the comparative statics on retail price. The six equations above from (1) can be reduced to two equations by substituting the first two equations into the third and fourth, and by equating these two equations with the last two equations:

$$\begin{aligned} p_f &= g\{uc_f(p_f, w)D[uc(p_f, w), z], c\} \\ w &= h\{uc_w(p_f, w)D[uc(p_f, w), z], t\} \end{aligned} \tag{2}$$

The comparative statics of these two equations can be characterized in matrix notation as follows:

$$\begin{bmatrix} \varepsilon_f + (1-s_f)\sigma - s_f e & -(1-s_f)(\sigma + e) \\ -s_f(\sigma + e) & \varepsilon_w + s_f\sigma - (1-s_f)e \end{bmatrix} \begin{bmatrix} d\log p_f \\ d\log w \end{bmatrix} = \begin{bmatrix} \varepsilon_f \varepsilon_c d\log c + e_z d\log z \\ \varepsilon_w \varepsilon_t d\log t + e_z d\log z \end{bmatrix}$$

where ε_f and ε_w are elasticites of farm input and marketing input with respect to their own prices; ε_c and ε_w are elasticites of farm price and marketing input price with respect to their respective supply shifters; and e_z is the elasticity of retail demand with respect to the retail demand shift variable. All the other variables were defined previously.

The solution to this system of equations can be represented as follows:

$$E_{p_f c} = \frac{\partial \log p_f}{\partial \log c} = \frac{[\varepsilon_w + s_f \sigma - (1 - s_f) e]\varepsilon_f \varepsilon_c}{|A|} \tag{3}$$

$$E_{p_f t} = \frac{\partial \log p_f}{\partial \log t} = \frac{(1 - s_f)(\sigma + e)\varepsilon_w \varepsilon_t}{|A|} \tag{4}$$

$$E_{p_f z} = \frac{\partial \log p_f}{\partial \log z} = \frac{(\varepsilon_w + \sigma) e_z}{|A|} \tag{5}$$

$$E_{wc} = \frac{\partial \log w}{\partial \log c} = \frac{s_f (\sigma + e)\varepsilon_f \varepsilon_c}{|A|} \tag{6}$$

$$E_{wt} = \frac{\partial \log w}{\partial \log t} = \frac{[\varepsilon_f + (1 - s_f)\sigma - s_f e]\varepsilon_w \varepsilon_t}{|A|} \tag{7}$$

$$E_{wz} = \frac{\partial \log w}{\partial \log z} = \frac{(\varepsilon_f + \sigma) e_z}{|A|} \tag{8}$$

where $|A| = [\varepsilon_f + (1 - s_f)\sigma - s_f e][\varepsilon_w + s_f \sigma - (1 - s_f)e] - s_f (1 - s_f)(\sigma + e)^2 > 0$.

The comparative statics of retail price can be obtained through differentiation of the second equation in (1), $p_r = uc(p_f, w)$, and substitute from equations (3)-(8) to obtain:

$$E_{p_r c} = \frac{\partial \log p_r}{\partial \log c} = \frac{s_f(\varepsilon_w + \sigma)\varepsilon_f \varepsilon_c}{|A|} \tag{9}$$

$$E_{p_r t} = \frac{\partial \log p_r}{\partial \log t} = \frac{(1-s_f)(\varepsilon_f + \sigma)\varepsilon_w \varepsilon_t}{|A|} \tag{10}$$

$$E_{p_r z} = \frac{\partial \log p_r}{\partial \log z} = \frac{[s_f \varepsilon_w + (1-s_f)\varepsilon_f + \sigma]e_z}{|A|} \tag{11}$$

Industry supply and industry derived demand for factors (Muth)

Alternative definitions of the retail-farm price spread:

(a) Retail-farm price ratio: p_r / p_f.

(b) Farm value share: s_f.

(c) Marketing margin:

a. $m = p_r - \alpha p_f$.

b. $m = p_r - (q_f / q_r) p_f$.

Determinants of the retail-farm price ratio (Gardner):

$$E_{p_r c} - E_{p_f c} = \frac{(1-s_f)(e - \varepsilon_w)\varepsilon_f \varepsilon_c}{|A|} \tag{12}$$

$$E_{p_r t} - E_{p_f t} = \frac{(1-s_f)(\varepsilon_f - e)\varepsilon_w \varepsilon_t}{|A|} \tag{13}$$

$$E_{p_r z} - E_{p_f z} = \frac{(1-s_f)(\varepsilon_f - \varepsilon_w)e_z}{|A|} \tag{14}$$

The impact of decreases in farm cost and increases in marketing input costs on the retail-farm price ratio are unambiguously positive as indicated by (12) and (13). However, the retail-farm price ratio will decrease (increase) depending upon whether the supply elasticity of the farm input is smaller (larger) than the elasticity of the marketing input.

Determinants of the farm value share:

Note $\log s_f = \log p_f - \log p_r + \log q_f - \log q_r = -\log(p_r / p_f) + \log(q_f / q_f)$. Also, with constant returns to scale, $\frac{\partial \log(q_f / q_r)}{\partial \log(p_r / p_f)} = \sigma$. This means that

$\frac{\partial \log s_f}{\partial \log x} = (-1+\sigma)\frac{\partial \log(p_r / p_f)}{\partial \log x}$, where x is an exogenous shift in the farm value share. Therefore, the determinants of the farm value share are simply equations (12)-(14) multiplied by $(-1+\sigma)$. If σ <1 then just the opposite qualitative results hold as in the case of the retail-farm price ratio. That is, an increase in farm costs or marketing input costs will cause the farm value share to increase and the farm value share will either decrease or increase with an increase in demand depending upon whether supply elasticity of the farm input is larger or smaller than the supply elasticity of the marketing input. On the other hand if σ >1, then the same results would hold as in the case of the retail-farm price ratio.

Determinants of the absolute Marketing Margin:

Case (a): $m = p_r - \alpha p_f$.

Let $s_f = \alpha(p_f / p_r)$. Upon totally differentiating this relationship we obtain

$$(m / p_r) d \log m = d \log p_r - s_F d \log p_f$$

$$= (E_{p_r c} - s_f E_{p_f c}) d \log c + (E_{p_r t} - s_f E_{p_f t}) d \log t + (E_{p_r z} - s_f E_{p_r z}) d \log z$$

Therefore,

$$E_{mc} = \frac{s_f(1-s_f)(\sigma+e)\varepsilon_f\varepsilon_c}{|A|(m/p_r)} \tag{15}$$

$$E_{mt} = \frac{(1-s_f)[(\varepsilon_f+\sigma)-s_f(\sigma+e)]\varepsilon_w\varepsilon_t}{|A|(m/p_r)} \tag{16}$$

$$E_{mz} = \frac{(1-s_f)(\varepsilon_f+\sigma)e_z}{|A|(m/p_r)} \tag{17}$$

Higher farm costs (reduced farm supply) leads to a lower margin, higher marketing costs lead to a higher margin as long as σ is small, and higher retail demand leads to a higher margin. What happens when ε_w approaches infinity?

Case (b): $m = p_r - (q_f/q_r)p_f$.

The total differential of this expression is

$$(m/p_r)d\log m = d\log p_r - s_f d\log p_f - s_f d\log(q_f/q_r)$$

Because with constant returns to scale, $\dfrac{\partial \log(q_f/q_r)}{\partial \log(p_r/p_f)} = \sigma$, the comparative statics of this margin relationship can be characterized as follows:

$$E_{mc} = \frac{s_f(1-s_f)[(\sigma+e)-\sigma(e-\varepsilon_w)]\varepsilon_f\varepsilon_c}{|A|(m/p_r)} \tag{18}$$

$$E_{mt} = \frac{(1-s_f)[(\varepsilon_f + \sigma) - s_f(\sigma + e) - s_f\sigma(\varepsilon_f - e)]\varepsilon_w\varepsilon_t}{|A|(m/p_r)} \tag{19}$$

$$E_{mz} = \frac{(1-s_f)[(\sigma + \varepsilon_f) - s_f\sigma(\varepsilon_f - \varepsilon_w)]e_z}{|A|(m/p_r)} \tag{20}$$

Note that as $\varepsilon_w \to \infty$ $\operatorname{sgn} E_{mc} = -\operatorname{sgn}(E_{p_r c} - E_{p_f c}), \operatorname{sgn} E_{mz} = -\operatorname{sgn}(E_{p_r z} - E_{p_f z})$.

Marketing Margins: Empirical Analysis

The general structural model underlying empirical analysis of price spreads for food products can be formulated as follows (Wohlgenant 2001, pp. 937-938):

$$\begin{aligned}
q_{rd} &= D_r(p_r, z) \\
q_{rs} &= S_r(p_r, p_f, w, t) \\
q_{fd} &= D_f(p_r, p_f, w, t) \\
q_{fs} &= S_f(p_f, c) \\
q_{rd} &= q_{rs} = q_r \\
q_{fd} &= q_{fs} = q_f
\end{aligned} \tag{1}$$

where it is assumed that w is exogenous. The first equation is the retail demand function, the second is the retail supply function, the third is the farm input demand function, the fourth is the farm input supply function, and the last two equations are market-clearing conditions in the retail and farm product markets. If we also assume that q_f is predetermined (e.g., lagged rather than current prices determine quantity supplied), then the market structure consists of the above equations less the fourth equation. Upon eliminating q_r from the equations above, the set of equations in (1) can be represented as follows:

$$\begin{aligned} &D_r(p_r, z) - S_r(p_r, p_f, w, t) = 0 \\ &q_f = D_f(p_r, p_f, w, t) \end{aligned} \tag{2}$$

The solutions to this set of equations are the reduced-form retail and farm price equations:

$$\begin{aligned} &p_r = p_r(z, w, t, q_f) \\ &p_f = p_f(z, w, t, q_f) \end{aligned} \tag{3}$$

The comparative statics of shifts in retail demand, marketing input prices, technology shifts (t), and farm input quantity on prices are obtained through differentiating the equations in (2). The reduced-form relations, (3), in log differential form are as follows:

$$\begin{aligned} &d\log p_r = \pi_{rz} d\log z + \pi_{rw} d\log w + \pi_{rt} d\log t + \pi_{rf} d\log q_f \\ &d\log p_f = \pi_{fz} d\log z + \pi_{fw} d\log w + \pi_{ft} d\log t + \pi_{ff} d\log q_f \end{aligned} \tag{4}$$

where

$$\begin{aligned} &\pi_{rz} = -\xi_{ff} / D, \\ &\pi_{rw} = (\xi_{ff}\xi_{rw} - \xi_{rf}\xi_{fw}) / D, \\ &\pi_{rt} = (\xi_{ff}\xi_{rt} - \xi_{rf}\xi_{ft}) / D, \\ &\pi_{rf} = \xi_{rf} / D, \\ &\pi_{fz} = \xi_{fr}\eta_z / D, \\ &\pi_{fw} = [\xi_{fr}\xi_{rw} + (\xi_{rr} - \eta)\xi_{fw}] / D, \\ &\pi_{ft} = [\xi_{fr}\xi_{rt} + (\xi_{rr} - \eta)\xi_{ft}] / D, \\ &\pi_{ff} = -(\xi_{rr} - \eta) / D, \\ &D = -(\xi_{rr} - \eta)\xi_{ff} + \xi_{rf}\xi_{fr}. \end{aligned}$$

What are the signs of the reduced-form elasticities in (4)?

(i) π_{ff} <0 because $E_{ff} = 1/\pi_{ff}$ is the industry derived demand elasticity for the farm input.

(ii) $\xi_{rr} > 0, \xi_{ff} < 0 \Rightarrow D > 0$ when $\eta < 0$.

(iii) $\operatorname{sgn} \pi_{rz} = \operatorname{sgn} \eta_z$, which would be expected to be positive when the shift in retail demand is interpreted as a horizontal increase in demand.

(iv) $\pi_{rf} < 0$ if the farm input is a normal factor.

(v) $\operatorname{sgn} \pi_{fz} = -\operatorname{sgn} \pi_{rf}$, because $\xi_{rf} = -s_f \xi_{fr}$.

(vi) the signs of $\pi_{rw}, \pi_{rt}, \pi_{fw}, \pi_{ft}$ are indeterminate because the signs of ξ_{fw}, ξ_{ft} are indeterminate.

Elasticity of price transmission: $E_{rf} = \pi_{rf} / \pi_{ff} = \dfrac{\xi_{rf}}{-(\xi_{rr} - \eta)}$.

Relationship between E_{ff} and E_{rf}: $E_{ff} = E_{ff}^c + E_{rf}\left(\dfrac{\xi_{fr}}{\xi_{rr}}\right)\eta = E_{ff}^c + E_{rf}\left(\dfrac{e_{mc,f}}{s_f}\right)\eta$, where

$E_{ff}^c = \xi_{ff} - \xi_{rf}\xi_{fr} / \xi_{rr}$, and $e_{mc,f} = \dfrac{-\xi_{rf}}{\xi_{rr}} = \dfrac{-\xi_{fr} s_f}{\xi_{rr}}$ is the elasticity of marginal cost with respect to a change in price of the farm input.

If the aggregate production function exhibits constant returns to scale (CRTS):

$$E_{ff} = E_{ff}^c + s_f \eta$$

Whether there is CRTS or not, it is clear that E_{ff} in general is not proportional to the elasticity of retail demand and that its absolute value is not necessarily smaller than the absolute value of the elasticity of retail demand.

Empirical restrictions on (4):

Assume $\eta_z = 1$, then

$$\pi_{rf} = -s_f \pi_{fz} \qquad \text{(symmetry)}$$

$$\left.\begin{aligned} \pi_{rz} + \pi_{rw} &= 1 \\ \pi_{fz} + \pi_{fw} &= 1 \end{aligned}\right. \qquad \text{(homogeneity)}$$

$$\begin{aligned} \pi_{rz} &= -\pi_{rf} \\ \pi_{fz} &= -\pi_{ff} \end{aligned} \qquad \text{(CRTS)}$$

$$\begin{aligned} \pi_{rw} &= 0 \\ E_{ff} &= 1/\pi_{ff} = s_f \eta \end{aligned} \qquad \text{(fixed proportions)}$$

In empirical work define

$$d\log z_i = \sum_{j \neq i} \eta_{ij} d\log p_j + \eta_{iy} d\log y + d\log pop$$

Using estimates of η_{ij}'s from previous studies, estimated demand shift variables, $\hat{z}_i$'s, can be constructed and used in regressions with the above parametric restrictions imposed.

Comparative statics of marketing margin:

Consider the following relationship between retail and farm prices:

$$m = p_r - (q_f / q_r) p_f = m(z, w, t, q_f), \qquad (5)$$

after substituting for the retail and farm price functions from (3). In general, the signs of the different demand and supply shifters are indeterminate. However, in the special case of CRTS it can be shown that $\partial m / \partial z > 0, \partial m / \partial q_f < 0$. The impact of w, t are indeterminate as in the comparative statics of retail and farm price. The significance of these comparative static results is that the empirical work is consistent with these predictions.

Empirical applications:

Wohlgenant applied this modeling approach to eight food commodities (beef and veal, pork, poultry, dairy, eggs, fruit, vegetables, and processed fruit & vegetables). The results are broadly consistent with the theory and there is strong evidence of input substitutability. Among other things, this suggests that traditional approach to modeling price spreads and derived demand elasticities, based on fixed input proportions, is much too restrictive and leads to substantial understatement of true derived demand elasticities.

Derived Demand Interrelationships:

It is possible to formulate systems of derived demand functions in much the same way as systems of consumer demand functions. This is particularly useful because in empirical work we are either (a) presented with data that are more appropriate for estimation of demand relationships at some level upstream from the retail level (e.g., wholesale level), or (b) interested in questions that are more appropriately answered within a derived demand framework (e.g., what is the effect of an increase in income on farm level prices?).

Heiner (1982, 1984), Braulke (1984), and Chavas and Cox (1997) develop a methodology that allows us to formulate and specify systems of industry derived demand functions. Let $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_m \end{bmatrix}, P = \begin{bmatrix} p_1 \\ p_2 \\ p_m \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ x_n \end{bmatrix}, W = \begin{bmatrix} w_1 \\ w_2 \\ w_n \end{bmatrix}$ be mx1 vectors of outputs and output prices, and nx1 vectors of inputs and input prices. Define the market equilibrium prices for all m outputs as solutions to the m set of market equilibrium conditions:

$$S_i[P(W,I),W] = D_i[P(W,I),I], i = 1,...,m$$

where S_i refers to market supply of the ith output, D_i refers to market demand for the ith output, $P(W, I)$ is the mx1 vector of output price functions, and I is the exogenous consumer demand shift variable (e.g., income).

Define the profit function for a given firm as

$$\pi(P,W) = \max_{y,x} \pi = P'Y - W'X$$

$$\text{subject to:} \qquad F(Y, X) = 0.$$

In order to obtain a system of derived demand functions, define the artificial function

$$Z(Y, X, W, I) = P'(W, I)Y - W'X - \pi[P(W, I), W]$$

For given Y, X :

$$\frac{\partial Z}{\partial w_i} = \sum_{k=1}^{m} \frac{\partial p_k}{\partial w_i} y_k - x_i - \sum_{k=1}^{m} \frac{\partial \pi}{\partial p_k} \frac{\partial p_k}{\partial w_i} - \frac{\partial \pi}{\partial w_i}, i = 1,..., n$$

For a maximum, the function $\frac{\partial Z}{\partial w_i} = 0 \Rightarrow \frac{\partial \pi}{\partial w_i} = -x_i^*(W, I)$. Also, for a maximum, the matrix, $[\partial^2 Z / \partial w_i \partial w_j]$ must be negative semi-definite.

The second-order partial derivatives of the function Z have the form:

$$\frac{\partial^2 Z}{\partial w_i \partial w_j} = -\sum_k \frac{\partial^2 \pi}{\partial w_i \partial p_k} \frac{\partial p_k}{\partial w_i} - \frac{\partial^2 \pi}{\partial w_i \partial w_j}$$

But $\sum_k \frac{\partial^2 \pi}{\partial w_i \partial p_k} = \frac{\partial y_k^*}{\partial w_i}$ and $\frac{\partial^2 \pi}{\partial w_i \partial w_j} = \frac{\partial x_i^*}{\partial w_j}$.

In matrix notation,

$$-\begin{bmatrix} y^*_{1w_1} & y^*_{2w_1} & y^*_{mw_1} \\ y^*_{1w_2} & y^*_{2w_2} & y^*_{mw_2} \\ y^*_{1w_n} & y^*_{2w_n} & y^*_{mw_n} \end{bmatrix} \begin{bmatrix} p_{1w_1} & p_{1w_2} & p_{1w_n} \\ p_{2w_1} & p_{2w_2} & p_{2w_n} \\ p_{mw_1} & p_{mw_2} & p_{mw_n} \end{bmatrix} + \begin{bmatrix} x^*_{1w_1} & x^*_{1w_2} & x^*_{1w_n} \\ x^*_{2w_1} & x^*_{2w_2} & x^*_{2w_n} \\ x^*_{nw_1} & x^*_{nw_2} & x^*_{nw_n} \end{bmatrix}$$

or $$\boldsymbol{Z}_{ww} = -\boldsymbol{Y}'^*_w \boldsymbol{P}_w + \boldsymbol{X}^*_w \qquad (1)$$

Differentiating each y^*_i with respect to w_j we find

$$Y'^*_w = P'_w D_p \qquad (2)$$

where $D_p = [\partial D_i / \partial p_k]$ is an mxm matrix of partial derivatives of the m final product demand functions with respect to the m product prices.

To establish the reciprocity conditions of the system of derived demand functions, post-multiply the matrix in (2) by P_w: $Y'^*_w P_w = P'_w D_p P_w \leq 0$, provided D_p is negative semi-definite (which will occur when it is the Slutsky substitution matrix). Note also that Z_{ww} is a negative semi-definite matrix (in order for Z to be a maximum). Therefore, upon combining (1) and (2) we have

$$X^*_w \leq Y'^*_w P_w = P'_w D_p P_w \leq 0 .$$

This means that the nxn matrix

$X^*_w = X^*_w(W, I)$ is negative semi-definite, implying that

$\partial x^*_i / \partial w_i \leq 0$ and $\partial x^*_i / \partial w_j = \partial x^*_j / \partial w_i$. It is also easy to show that the derived demand functions are homogenous of degree zero in W, I .

Applications:

(a) Specification and estimation of systems of derived demand functions.

(b) Evaluation of validity of traditional derived demand formulas.

<u>Impact of Imperfect Competition:</u>

Consider Schroeter's (1988) specification of processor behavior in the beef processing industry. Each firm (denoted as j) is assumed to maximize the profit function:

$$\pi^j = p_r q_r^j - p_f q_f^j - c^j(q_r^j, \mathbf{w}) = p_r q^j - p_f q^j - C^j(q^j, \mathbf{w})$$

where it is assumed that $q_r^j = q_f^j = q^j$. The F.O.N.C. for profit maximization is

$$\frac{\partial \pi^j}{\partial q^j} = p_r + \frac{\partial p_r}{\partial q}\frac{\partial q}{\partial q^j} q^j - p_f - \frac{\partial p_f}{\partial q}\frac{\partial q}{\partial q^j} q^j - \frac{\partial C^j}{\partial q^j} = 0 \qquad (1)$$

where $p_r = h(q_r, z_1) = h(q, z_1), q = \sum_j q^j$, is the industry inverse demand function for the retail product, and $p_f = g(q_f, z_2) = g(q, z_2)$ is the industry inverse supply function for the farm input. Let η, ε be the industry elasticity of retail demand and industry elasticity of farm supply, respectively; and let $\theta^j = (\partial q / \partial q^j)(q^j / q)$ be the firm's perceived rate of change in market output (and farm input) with respect to the firm's own output (and farm input quantity). For price-taking firms, $\theta^j = 0$; for monopolists, $\theta^j = 1$. In the majority of cases, we would expect θ^j to lie between zero and one.

If use is made of industry-wide data, we need to aggregate (1) to its industry-wide counterpart. Recall that the Gorman polar form of the cost function is required for consistent aggregation across the economic agents. Therefore, in this case, specify the cost function as

$$C^j(q^j, \mathbf{w}) = c^j(\mathbf{w}) q^j + g^j(\mathbf{w})$$

Therefore, upon aggregation over all firms in the industry, we have

$$C(q,\mathbf{w}) = \sum_j C^j(q^j,\mathbf{w}) = c(\mathbf{w})\sum_j q^j + \sum_j g^j(\mathbf{w}) = c(\mathbf{w})q + \sum_j g^j(\mathbf{w}) \qquad (2)$$

Therefore, upon aggregating over firms, equation (1) can be represented as follows:

$$p_r\left(1+\frac{\theta}{\eta}\right) - p_f\left(1+\frac{\theta}{\varepsilon}\right) = c(\mathbf{w}) \qquad (3)$$

Equation (3) together with specifications for the cost function (2), and industry output and input supply can be estimated jointly to determine magnitude of market power. The degree of price distortion can be measured as follows:

$$L = (p_r - mc)p_r = \theta/\eta,$$
$$M = (mnrp - p_f)/p_f = \theta/\varepsilon$$

Schroeter (1988) applied this model to the U.S. beef processing industry and found small, but significant, market power. Values for L and M ranged from between 1% and 3% in later years of the sample period. (Model was estimated with data over the time period 1951-1983).

Holloway (1991) modified the framework of Wohlgenant (1989) to test directly for monopoly power. His test for market power amounts to a test on the retail-to-farm price ratio. He exploits the fact that, under perfect competition, shifts in retail demand and farm supply are equal in magnitude but opposite in sign. From the relationship between retail and farm prices for Wohlgenant's (1989) model we have

$$d\log(p_r/p_f) = (\pi_{rz} - \pi_{fz})d\log z + (\pi_{rw} - \pi_{fw})d\log w + (\pi_{rf} - \pi_{ff})d\log q_f$$

where for the sake of simplicity we have omitted the term *t* from the equation. Given this specification, the test for perfect competition is then

$$H_0 : \pi_{rz} - \pi_{fz} = -(\pi_{rf} - \pi_{ff})$$

But if $\pi_{rz} = -\pi_{rf}$ and $\pi_{fz} = -\pi_{ff}$, then the above conditions hold automatically. So Holloway's test is equivalent to the restriction of CRTS derived by Wohlgenant (1989).

Problems with Schoeter and Holloway models:

1. Assumption of fixed proportions in Schroeter model may bias results toward rejection of competition. To accommodate variable proportions, we encounter an identification problem. Output demand must be non-separable in at least one of the exogenous shifters of demand in order to identify the degree of market power. The same holds true for the farm input supply curve.
2. Holloway's test for market power is really a joint test for CRTS and competition. If there is decreasing returns to scale or increasing returns to scale, it is unclear what we are testing.

General Approach (Muth and Wohlgenant 1999):

Define $\pi = \pi(\tilde{p}_r, \tilde{p}_f, w), \tilde{p}_r = p_r(1+L), \tilde{p}_f = p_f(1+M)$. The supply and demand *relations* are obtained as

$$\frac{\partial \pi}{\partial \tilde{p}_r} = S_r(\tilde{p}_r, \tilde{p}_f, w)$$
$$\frac{\partial \pi}{\partial \tilde{p}_f} = -D_f(\tilde{p}_r, \tilde{p}_f, w)$$

Because $\frac{\partial S_r}{\partial \tilde{p}_f} = -\frac{\partial D_f}{\partial \tilde{p}_r}$ and $\frac{\partial S_r}{\partial p_f} = (1+M)\frac{\partial S_r}{\partial \tilde{p}_f}, \frac{\partial D_f}{\partial p_r} = (1+L)\frac{\partial D_f}{\partial \tilde{p}_r}$,

$$(1+M)\frac{\partial S_r}{\partial \tilde{p}_f} = -(1+L)\frac{\partial D_f}{\partial \tilde{p}_r}$$

Therefore, a test for market power is equivalent to a test for price taking behavior, that is

$$H_0 : M = L = 0 .$$

Again, however, we may run into identification problems unless elasticities of demand and supply and/or conjectural elasticities are allowed to change over time with changes in market conditions. However, the framework can still be useful for testing for the presence of market power.

General observations on testing for market power:

1) We need to be careful not to associate concentration ratios with market structure. That is, the number of firms may have little to do with degree of competitiveness. We need to make a distinction between competition *in* the market and competition *for* the market. In other words, we need to account for the effects of *potential* entry on the market. In such cases, the "invisible hand" of competition becomes even more constraining on firms' behavior and something approximately equal to perfectly competitive equilibrium is more likely to emerge.

2) We need to account for cost savings from increased concentration (mergers), the effects of which can partially or totally offset any effects of anti-competitive behavior.

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VII. Dynamic Factor Demand

Dynamic Factor Demand with Capital

Suppose the market middlemen procures raw materials and other variable factors, $\mathbf{x}$, at prices, $\mathbf{w}$, and combines with capital, k, to produce current output, q, and future net output, k', according to the production function, $q = f(\mathbf{x}, k, k')$. The firm selects the variable factors, given k, k' to obtain the short-run profit function, $\pi(p, \mathbf{w}, k, k')$, where p is output price.

Suppose that the capital stock evolves over time according to the equation

$$k' = i - \delta k, \qquad k(0) = k_0 \tag{1}$$

The firm's problem is to select time paths of capital and investment, i, in order to maximize the present discounted value of future expected profit

$$\int_0^\infty e^{-rt} \, [\pi(p, \mathbf{w}, k, k') - qi] dt \tag{2}$$

subject to the constraint in (1). The current value Hamiltonian associated with this problem is

$$H = \pi(p, \mathbf{w}, k, k') - qi + \lambda(i - \delta k) = \pi(p, \mathbf{w}, k, i - \delta k) - qi + \lambda(i - \delta k)$$

(3)

The first-order necessary conditions associated with maximizing the function (3) include

$$\frac{\partial H}{\partial i} = \frac{\partial \pi}{\partial k'} - q + \lambda = 0$$

$$\frac{\partial H}{\partial k} = \frac{\partial \pi}{\partial k} - \frac{\partial \pi}{\partial k'}\delta - \delta\lambda = r\lambda - \lambda'$$

(4)

given the side conditions (1) and the transversality condition

$$\lim_{t \to \infty} e^{-rt}\lambda = 0 \qquad (5)$$

If the Hamiltonian, equation (3), is a concave function in k, i, the necessary conditions will also be sufficient. Assuming the unit cost of investment, q, is viewed as constant over the planning horizon, the conditions in (4) can be reduced to the single equation:

$$\pi_{k'k'}k'' + \pi_{k'k}k' = \pi_k + r\pi_{k'} - (r + \delta)q \qquad (6)$$

where subscripts denote first and second order partial derivatives. The steady-state solution to (6) is

$$\pi_k(k^*, k' = 0) + r\pi_{k'}(k^*, k' = 0) = (r + \delta)q \qquad (7)$$

Linearize (6) around the point $k = k^*, k' = 0$:

$$\pi_{k'k'}k'' + (\pi_{k'k} - \pi_{kk'} - r\pi_{k'k'})k' - (\pi_{kk} + r\pi_{k'k})(k - k^*) = 0 \tag{8}$$

This second-order homogenous differential equation has the form

$$am^2 + bm + c = 0$$

which has solutions: $m = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Choose the smallest root to get

$$k' = m(k^* - k) \tag{9}$$

where $m = \dfrac{-\left(r - \sqrt{r^2 + 4(\pi_{kk} / \pi_{k'k'})}\right)}{2}$.

Equations to estimate econometrically include equation (9) and

$$\begin{aligned} q &= \frac{\partial \pi(p, \mathbf{w}, k, k')}{\partial p} \\ x_i &= -\frac{\partial \pi(p, \mathbf{w}, k, k')}{\partial w_i} \end{aligned} \tag{10}$$

Equation (9) is the (net) investment function and the equations in (10) are the short-run supply and short-run input demand functions for variable inputs.

Lopez postulates normalized quadratic function for profit function. Given this functional form, long-run optimal capital stock, k^*, can be derived from equation (7). He applies the model to the Canadian food processing industry where inputs are labor, energy, and intermediate inputs (95% of which is raw food input). He estimates supply and input demand functions for different lengths of run. He finds that the short-run output supply and input demand elasticities obtained through differentiation of the equations in (10).

Long-run supply and demand functions are obtained through differentiation of solution to equation (7) and differentiation of equations (10) when k' =0.

Multivariate Extension (Morrison Paul)

Short-run Lags in Price Spreads

Wohlgenant (1985) develops model to explain how inventory holding can explain short-run lags in adjustment of retail price to changes in wholesale price. The idea is that the unit cost function often used to specify short-run costs is misspecified. Normally, the firm is hypothesized to produce a retail product using a fixed proportions technology so that the firm's price equals unit raw material costs plus unit processing/marketing costs. For example, Heien specifies that

$$P_t = W_t + aC_t$$

where P_t is retail price, W_t is the wholesale price in units of the retail product, and C_t represents processing/marketing input prices like the wage rate, energy prices, etc. The misspecification arises from the fact that if inventories are held the firm incurs storage costs per unit time which are typically proportional to the price of the wholesale product. The above price equation should be augemented with these costs and can be written as

$$\overline{P_t} = W_t + aC_t + g(1-b)W_t$$

where the $\overline{P_t}$ now represents the steady-state price level and $g(1-b)W_t$ represent unit storage costs. The parameters g and b reflect the average length of time in storage and interest costs on money spent on inventories foregone (i.e., opportunity cost of short-run capital). In the short run the actual price P_t can deviate from its steady-state value, $\overline{P_t}$, based on whether the firm expects to incur capital gains or capital losses on holding stocks:

$$P_t = \overline{P_t} + bg(W_t - E_tW_{t+1})$$

$$= W_t + aC_t + g(1-b)W_t + bg(W_t - E_t W_{t+1})$$
$$= W_t + aC_t + bg[(b^{-1}-1)W_t + (W_t - E_t W_{t+1})]$$

Because $= 1/(1+r)$, where *r* is the real interest rate, the expression for the price equation becomes:

$$P_t = W_t + aC_t + bg[rW_t + (W_t - E_t W_{t+1})] \qquad (1)$$

In this form, the price equation shows that, intuitively, retail price equals unit raw material cost plus unit processing/marketing costs plus the user cost of inventory capital. In the steady-state when $(W_t - E_t W_{t+1}) = 0$, retail price equals unit raw material, processing/marketing plus the interest cost on holding inventories (average length of time held in inventory, g, times (discounted) interest rate, *br*.

If capital gains are expected from holding inventory, $(W_t - E_t W_{t+1}) < 0$ (i.e., the raw material price is expected to increase in the future), the firm's opportunity cost of holding inventories declines , which in turn, induces it to hold more inventories. This also means that the firm will sell more of the retail product now than later, the effect of which will be to cause retail price to be below its steady-state: $P_t < \overline{P_t}$.

Likewise, if capital losses are expected from holding inventory, $(W_t - E_t W_{t+1}) > 0$ (i.e., the raw material price is expected to decrease in the future), the opportunity cost of holding inventories increases, which in turn, induces it to hold less inventories. This also implies that the firm will sell less now compared to later, the effect of which will be to cause retail price to be above its steady-state: $P_t > \overline{P_t}$.

Whether capital gains or losses are anticipated, it seems clear that at any point in time the firm is likely to find itself in one position or another and to adjust its inventories respectively. In either case, because of transaction and information costs we would not expect the firm to instantly adjust inventories to their steady-state level. The effect of this will be to see a time lag between expected changes in the price of the raw material and changes in retail price. Equivalently, this model explains why retail price changes lag wholesale price changes.

Note that nothing has been said whether the firm is a price taker or has some marketpower. Indeed, all the above can be shown to follow from just price taking behavioronthe part of individual firms. In fact, the equilibrium price relationship can be rewritten to show that

$$\bar{P}_t = [1 + g(1-b)]W_t + aC_t \qquad (2)$$

The equation now shows that the firm does engage in markup pricing, but the markup is due to inventories not market power. The estimated coefficients when applied to the beef market indicate that $\hat{b}$ = .89and $\hat{g}$ = 1.97 so that the markup is

$$[1 + g(1-b)] = 1 + 1.97(1 - .89) = 1.22$$

The markup of 1.22 is in line with much of the estimates in the literature for food commodities which most authors attribute to market power. However, all the effect here has been attributed to inventories!

The above equation can be rigorously derived from an intertemporal model of the firm where the firm is assumed to maximize the present discounted value of profits

$$E_t \sum_{j=0}^{\infty} b^j \left[(P_{t+j} - aC_{t+j})q_{t+j} - W_{t+j}q_{t+j} - \frac{f}{2}\left(i_{t+1+j} - g_0 - gs_{t+j}\right)^2 \right]$$

subject to the constraint: $i_{t+1+j} = i_{t+j} + q_{t+j} - q_{t+j}$. The firm is assumed to formulate such a problem each real time period *t*, which means it only acts on the first period of its planned problem *j*=0.

Other inventory models: newsvendor problem.

Pelzman—why do prices rise faster than they fall?

Inventories and Futures Markets: Brennan, Williams and Wright

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VIII. Dynamic Consumer Demand

Consumers very often react with delay to price and income changes with implication that adjustment to new equilibrium is spread over several time periods.

$$q_t = a + b_0 p_t + b_1 p_{t-1} + \ldots + c_0 x_t + c_1 x_{t-1} + \ldots + \varepsilon_t$$

$$= a + \sum_{i=0}^{\infty} b_i p_{t-i} + \sum_{i=0}^{\infty} c_i x_{t-i} + \varepsilon_t \qquad (1)$$

What are the origins of these lags?

(a) Durabilility of good: individual purchases a stock (e.g., a car) but consumes a flow of services (e.g., mileage, transportation) over time. As income increases,there will be an increased demand for flow from given stock. Ultimately, this will lead to an increase in the stock level, but only after enough time has elapsed so the consumer is sure income change is permanent.

(b) Habit formation: after a price increase for which the consumer has developed habits, consumer may appear to buy quantities which are larger than equilibrium values indicated by static demand function. At first sight, the consumer is out of equilibrium. One may then imagine a slow adaptive process by which he/she gradually reduces consumption to the “correct” level.

Alternative models:

Estimation of equation (1) is usually not possible unless some restrictions are placed on the lag distributions of prices and income.

Geometric distribution:

Suppose that

$$q_t = a + b\sum_{i=0}^{\infty}\theta_i p_{t-i} + cx_t + \varepsilon_t \tag{2}$$

where $\sum_{i=0}^{\infty}\theta_i = 1$. Assume that $\theta_i = \delta(1-\delta)^i$ then

$$q_t = a + b\delta\sum_{i=0}^{\infty}(1-\delta)^i p_{t-i} + cx_t + \varepsilon_t \tag{3}$$

Because $\sum_{i=0}^{\infty}(1-\delta)^i p_{t-i} = \frac{1}{1-(1-\delta)L} p_t$, equation (3) can be re-written as

$$\begin{aligned} q_t &= a + (1-\delta)q_{t-1} + b\delta p_t + cx_t - c(1-\delta)X_{t-1} + \varepsilon_t^* \\ \varepsilon_t^* &= \varepsilon_t - (1-\delta)\varepsilon_{t-1} \end{aligned} \tag{4}$$

Partial adjustment model:

$$q_t^* = a + bp_t + cx_t + \varepsilon_t$$

$$q_t - q_{t-1} = \delta(q_t^* - q_{t-1})$$

The first equation is long-run demand and the second equation shows how demand adjusts to its long-run equilibrium level where the parameter δ is assumed to lie between zero and one. Upon substituting the unobserved long-run demand function into the second equation we obtain the partial adjustment demand model:

$$q_t = \delta a + \delta b p_t + \delta c x_t + (1-\delta) q_{t-1} + \delta \varepsilon_t \tag{5}$$

State adjustment model:

In contrast to the partial adjustment model, which starts with long-run demand, this model starts with short-run demand

$$q(t) = \theta + \alpha s(t) + \beta p(t) + \gamma x(t) \tag{6}$$

where $s(t)$ is the "state variable" . For durable goods, this variable represents physical stocks; for nondurables, this variable represents stock of habits. For stock adjustment, we would expect the parameter α <0; for habit formation, we would expect α >0. The state variable is assumed to evolve according to the state equation

$$s'(t) = q(t) - \delta s(t) \tag{7}$$

In general, the state variable, $s(t)$, is unobservable and can be eliminated by using both equations (6) and (7) to obtain

$$q'(t) = \delta\theta + (\alpha - \delta) q(t) + \beta p(t) + \delta\beta p'(t) + \gamma x(t) + \delta\gamma x'(t) \tag{8}$$

In long-run equilibrium $s'(t) = 0 \Rightarrow p'(t) = x'(t) = q'(t) = 0$, so from (8) we have

$$q^* = \frac{\delta\theta + \beta\delta p^* + \gamma\delta x^*}{(\delta - \alpha)} \tag{9}$$

provided $(\delta - \alpha) \neq 0$.

For empirical estimation, the model needs to be converted to discrete time which necessitates a side assumption be made about how the state variable changes over the discrete time interval. If it evolves linearly within each time interval, then equations (6) and (7) are replaced by the equations

$$q_t = \theta + \alpha\left(\frac{s_t + s_{t-1}}{2}\right) + \beta p_t + \gamma x_t$$

$$\Delta s_t = s_t - s_{t-1} = q_t - \delta\left(\frac{s_t + s_{t-1}}{2}\right)$$

Following the procedure before to eliminate the unobservable state variable we would obtain the estimable form of the state adjustment model:

$$q_t = A_0 + A_1 q_{t-1} + A_2 \Delta p_t + A_3 p_{t-1} + A_4 \Delta x_t + A_5 x_{t-1} \tag{10}$$

where the A_i 's are functions of the underlying structural parameters. It can be shown that these reduced-form parameters are over-identified but can be just identified if the following restriction holds:

$$\frac{A_2}{A_3} = \frac{(1+\delta/2)}{\delta} = \frac{A_4}{A_5} \tag{11}$$

Therefore, with an error term appended to equation (10), the estimation problem is to estimate equation (10) subject to the non-linear restriction on the parameters given by equation (11).

Dynamic LES:

$$p_i q_i = p_i \gamma_i + \beta_i (x - \sum_j p_j \gamma_j)$$

$$\gamma_i = \theta_i + \alpha_i s_i \tag{12}$$

where the state variable, as before, is assumed to evolve according to the equation (7). The long-run demand functions corresponding to the short-run demand functions in (12) are

$$p_i q_i = p_i \gamma_i^* + \beta_i^* (x - \sum_j p_j \gamma_j)$$

$$\gamma_i^* = \frac{\gamma_i \theta_i}{(\delta_i - \alpha_i)}, \beta_i^* = \frac{\delta_i \beta_i /(\delta_i - \alpha_i)}{\sum_j \delta_j \beta_j /(\delta_i - \alpha_i)} \tag{13}$$

As before, when α_i >0, the commodity is habit forming; when α_i <0, the commodity is durable.

For empirical analysis, the first-order conditions for utililty maximization of the augmented Stone-Geary utility function are estimated. The equations can be shown to have the following form:

$$q_t = K_{i0} + K_{i1} q_{it-1} + K_{i2} \frac{1}{\lambda_t p_{it}} + K_{i3} \frac{1}{\lambda_{t-1} p_{it-1}}$$

subject to the budget constraint $x_t = \sum_j p_{jt} q_{jt}$.

Intertemporal analysis (Becker et al.):

It is sometimes necessary to have a model to account for the fact that consumers may anticipate expected future consequences of their current actions. Let c_t be the quantity of the good consumed in period t, y_t consumption of a composite commodity, and e_t the unmeasured life-cycle variables on utility. The consumer is assumed to maximize the present discounted value of future utility

$$\sum_{t=1}^{\infty} \beta^{t-1} u(c_t, c_{t-1}, y_t, e_t)$$

subject to

$$c_0 = c^0$$

$$\sum_{t=1}^{\infty} \beta^{t-1}(y_t + p_t c_t) = A^0$$

The F.O.N.C. for utility maximization are as follows:

$$u_y(\, c_t, c_{t-1}, y_t, e_t) = \lambda \tag{1}$$

$$u_1(\, c_t, c_{t-1}, y_t, e_t) + \beta u_2(c_{t+1}, c_t, y_{t+1}, e_{t+1}) = \lambda_t\, p_t \tag{2}$$

Condition (1) says that the marginal utility of other goods equals the marginal utility of wealth and condition (2) says that the marginal utility of current consumption plus discounted value of next period's marginal utility equals price times the marginal utility of wealth.

If the utility function is quadratic, then the F.O.N.C. in (1) and (2) are linear. After solving (1) for y_t and y_{t+1} and eliminating these two variables from (2) we get

$$c_t = \theta c_{t-1} + \beta\theta c_{t+1} + \theta_1 p_t + \theta_2 e_t + \theta_3 e_{t+1} \tag{3}$$

The coefficient θ_1<0 by the concavity of the utility function and θ>0 if the marginal utility of current consumption increases with increases in future consumption. (i.e., good is addictive).

Equation (3) forms the basis for empirical analysis. Note that ordinary least squares will yield inconsistent estimates because the variables e_t and e_{t+1} are correlated with c_{t-1} and c_{t+1}. Therefore, past and future prices are logical instruments for lagged and future consumption. Estimates of the parameters of equation (3) can be used to derive price elasticities for different lengths of run.

<u>Extension to Multivariate Case (Wohlgenant)</u>

<u>Simple Non-Additive Preferences (SNAP) Model</u>
Concept of Profit Function
Use of Profit Function to Specify Intertemporal Models
SNAP AIDS Model
A convenient framework to use to represent rational addiction is the profit function:

$$\pi(P^1, P^2, P^T, r) = \max_q \left[rU - \sum_t P^t \cdot q^t \right]$$

where P^t is the price vector of goods in period t, q^t is the quantity vector of goods in period t, U is the intertemporal utility function, and r is the reciprocal of the marginal utility of wealth. The demand functions are obtained simply by Roy's identity:

$$q^t = -\frac{\partial \pi(P^1, P^2, P^T, r)}{\partial q^t}$$

These demand functions are homogenous of degree zero in (P^1, P^2, P^T, r) and symmetric with respect to prices:

$$\frac{\partial q_i^t}{\partial p_j^s} = \frac{\partial q_j^s}{\partial p_i^t}$$

for all cross-price combinations of i, j, s, t.

To come up with tractable form of Frisch demand functions, use cost function/profit function duality relationship (Cooper 1994)

$$C^F(r, P^1, P^2, P^T) = r\frac{\partial \pi}{\partial r} - \pi$$

Solve for $r = r(P^1, P^2, P^T, x)$. Substituting into Frisch demand functions then produces ordinary or Marshallian demand functions conditional on definition of expenditure. The reason we only need to use one variable for r is because of the theoretical result stated earlier that the consumer desires to choose consumption over time so that the marginal utility of wealth or income is constant across time periods.

Browning proposes the model with simple non-separable preferences (SNAP)

$$\pi(P^1, P^2, P^T, r) = -\sum_{t=1}^{T-1} \Phi^t(\, P^t,\ P^{t+1}, r)$$

from which we get the Frisch demand functions

$$q^t = \frac{\partial \Phi^{t-1}(P^{t-1}, P^t, r)}{\partial P^t} + \frac{\partial \Phi^t(P^t, P^{t+1}, r)}{\partial P^t}$$

Browning uses the PIGLOG cost function to specify the model:

$$w_{it} = \alpha_i + \sum_j \gamma_{ij} \ln p_{jt} + \beta_i \ln\left[\frac{x_t}{P_t}\right] + \beta_i \sum_j \theta_j \ln p_{jt-1} - \theta_i \left[\frac{b(\mathbf{p}_t)}{b(\mathbf{p}_{t+1}^e)}\right]$$

where θ are parameters; $\mathbf{p}_{t+1}^{e}$ is the vector of expected beverage prices in period t+1 as of period t; and

$$\ln b(\mathbf{p}_t) = \sum_j \beta_j \ln p_{jt} \, .$$

Browning showed that forward-looking dynamic household behavior such as rational habit persistence or stockpiling can be conveniently tested as parametric restrictions on this form.

Empirical Dynamic Demand Systems (Anderson and Blundell):
Consumers are unlikely to adjust fully to price and income changes in every time period. Habit persistence, adjustment costs, incorrect expectations, and misinterpreted real price changes are potential causes. Therefore, a dynamic modeling approach is chosen.

$$w = f(z, \theta) \tag{1}$$

where $w = nx1$ vector of budget shares, $z = kx1$ vector of income and price variables, $\theta =$ underlying preference parameters. Let $f(z, \theta)$ be the AIDS specification then

$$f(z, \theta) = \pi(\theta) x \tag{2}$$

where $x = lx1$ vector of transformed income and price variables and $\pi(\theta)$ is a constant matrix function of θ.

A general first-order dynamic model may be written as

$$\Delta w_t = A^* \Delta x_t - B^* [w_{t-1} - \pi(\theta) x_{t-1}] + \varepsilon_t \tag{3}$$

which is equivalent to

$$w_t = A_1 x_t + A_2 x_{t-1} + C w_{t-1} + \varepsilon_t \tag{4}$$

where $B^* = (I - C)$, $A^* = A_1$, and $\pi(\theta) = B^{*-1}(A_1 + A_2)$. The advantage of (3) over (4) is in terms of direct parameterization of the long-run structure of $\pi(\theta)$.

Because (2) and (3) are systems of equations, the summation of the dependent variables equals unity and the system is singular. Therefore, the estimable form of (3) is

$$\Delta w_t = A\Delta\tilde{x}_t - B[w_{t-1} - \pi^n(\theta)x_{t-1}] + \varepsilon_t \tag{5}$$

where $\tilde{x}_t$ refers to the vector with the constant term excluded and the matrix B has the form

$$b_{ij} = b^*_{ij} - b^*_{in}, \qquad \forall i = 1,\dots,n;\ j = 1,\dots,n-1.$$

Without a priori restrictions, there is loss of identification since b^*_{ij} cannot be recovered from the estimated b_{ij}. However, the adding-up properties of $\pi(\theta)$ imply that there is no loss of identification in the long-run structure since all the underlying preference parameters θ can be derived from restrictions on π^n.

Models (3) and (4) include several interesting special cases:

(a) Autoregressive model: $A_2 = CA_1$.

(b) Partial adjustment: $A_2 = 0$.

(c) Static model: $A_2 = 0, C = 0$.

Cointegrating Demand Systems (Attfield):

Write system of demand functions in triangular ECM (TECM) framework:

$$y_{1t} = By_{2t} + u_{1t}$$

$$\Delta y_{2t} = C_1 \Delta y_{2t-1} + C_2 \Delta y_{2t-2} + \cdots + C_k \Delta y_{2t-k} + u_{2t}$$

where y_{1t} is the vector of expenditure shares of the LA/AIDS model, and $y_{2t}$is the vector of log prices and log real income. In this framework, FIML of the TECM system is equivalent to estimating

$$y_{1t} = B y_{2t} + C_0 \Delta y_{2t} + C_1 \Delta y_{2t-1} + \cdots + C_k \Delta y_{2t-k} + v_t$$

where the parameter matrices $C_1, C_2, \ldots, C_k$ are considered nuisance parameters. Econometric estimates when the variables are I(1) are superconsistent and standard errors approach asymptotically consistent values rapidly as lagged first differences are added to the model.

From an economic point of view, this approach produces long-run, or steady-state, elasticities that often satisfy homogeneity and symmetry restrictions. The motivation for this approach is unit roots in the time series variables.

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